GENERAL EQUATION FOR DIRECTED ELECTROMAGNETIC WAVE PROPAGATION IN 1D METAMATERIAL: PROJECTING OPERATOR METHOD

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Abstract: We consider a boundary regime problem for 1D wave propagation in a metamaterial medium with simultaneously negative dielectric permittivity and magnetic permeability. We apply a projecting operator method to the Maxwell system in the time domain that allows the space of the linear propagation problem to be split into subspaces of directed waves for the relations of a given material with general dispersion. After projection, the equations for directed waves have a maximally simplified form, which is most convenient for numerical and analytical integration. Matrix elements of the projectors act as integral operators.

For a given nonlinearity and dispersion we derive a general system of interacting right/left waves with combined (hybrid) amplitudes. The result is specified for the Drude metamaterial model for both permittivity and permeability coefficients and the Kerr nonlinearity. We also discuss and investigate singular solitary wave solutions of the system as a limit stationary elliptic system related to some boundary regimes.

Keywords: metamaterials, directed waves interactions, Drude dispersion, Kerr nonlinearity, solitary solutions

1. Introduction

The history of research on metamaterials starts from J. C. Bose's work [1]. He studied the rotation of plane of polarisation of electric waves by an artificial "chiral" structure created by him. One of the intriguing problems for researchers is related to artificial materials named later metamaterials that are characterized by both negative dielectric permittivity and magnetic permeability. The ideas connected with the negative refraction index and media appeared in the 1940s–1950s, and were described in the works of L. I. Mandelshtam [2], G. D. Malyuzhinets [3] and others. In 1968 Victor Veselago [4] wrote about the general electrodynamic properties of metamaterials.

In 2000 David Smith and his group created such a type of structures [5]. Structures with simultaneously negative dielectric permittivity and magnetic permeability have been called by many names: Veselago media, negative-refraction media, backward wave media, double-negative media, etc. [6]. The applications for metamaterials are broad and varied from the celebrated electromagnetic cloaking [7], to new imaging capabilities [8]. A practical implementation of a subwavelength resonator is presented in [9]. The idea for dispersion compensation in transmission lines using negative-refractive media (NRM) was described in [10]. An interaction of ultrashort pulses with ordinary materials is well understood in nonlinear optics [11] and extended for metamaterials in [12].

A problem of a derivation, or, bettyer to say – embedding of optical pulses propagation, directed by its physical sense, need elaboration of special methodics of Maxwell equations simplification [13]. An element of such method contains a transition to new variables, *e.g.* of the form

$$\psi^{\pm} = \varepsilon\frac{1}{2}E_i \pm \mu\frac{1}{2}H_j \tag{1}$$

as did Fleck [14], Kinsler *et al.* [15] and Amiranashvili [16] in their works. Other parts of the construction imply a combination of equations with strong account expansion by small parameters. Some general tool of this kind is based on a splitting of the evolution space into subspaces of roots of the corresponding dispersion equation. It in a sense "diagonalizes" the basic Maxwell system that leads to a set of equations of the first order in time that naturally include unidirectional equations. A natural construction is realized by a complete set of projecting operators, each for a given dispersion relation, for a linearized fundamental system [13, 17]. The method differs from those of [14, 15], as it allows us to combine *equations* of the complex basic system in an algorithmic way even with, after some development, nonlinearity taken into account, and simultaneously, introduces combined fields named basic modes. It therefore allows us to effectively formulate a corresponding mathematical problem: initial or boundary conditions in an appropriate physical language in a mathematically correct form. The method has many mutual features with the method of [18]

In this paper we apply the approach of projecting operators to the problem of wave propagation in a 1D-metamaterial with general dispersion originated from both material relations and nonlinearity. The main exposition of the work is very similar to [19]: we want to derive a general evolution equation for the mentioned conditions with minimal simplifications. The methodical differences and results are highlighted and discussed.

The article consists of an introduction, three sections and conclusion.

In the introduction the currency of the problem and basic ideas of the projection method are outlined.

In Section 2 we state the problem. We also show, how the material relations change when dispersion is taken into account.

Section 3 is devoted to projecting the operator construction in ω and t representations (domains).

Section 4 realizes the main task of the solutions space separation.

In Section 5 we account for nonlinearity and realize the important example of the Kerr nonlinearity, deriving a system of directed wave interaction.

Numbers 6 and 7 include realization of the program for Drude dispersion (5) and Kerr nonlinearity (6) models, and hence, finalize the main result of the paper: a general system of directed wave interaction for this model, popular in investigations of metamaterials. Section 7 includes also a subsection about stationary solutions that show the difference between conventional and Veselago materials.

2. Statement of problem

Our starting point are the Maxwell equations for a simple case of linear isotropic but dispersive dielectric media, in the SI unit system:

$$\operatorname{div}\vec{D}(\vec{r},t)=0 \qquad \operatorname{div}\vec{B}(\vec{r},t)=0 \tag{2}$$

$$\operatorname{rot}\vec{E}(\vec{r},t)=-\frac{\partial\vec{B}(\vec{r},t)}{\partial t} \qquad \operatorname{rot}\vec{H}(\vec{r},t)=\frac{\partial\vec{D}(\vec{r},t)}{\partial t} \tag{3}$$

Next we will write the derivatives as:

$$\partial_t\equiv\frac{\partial}{\partial t},\quad \partial_x\equiv\frac{\partial}{\partial x} \tag{4}$$

We restrict ourselves to a one-dimensional model, similarly to the publication [11], developing the results of Kuszner, Leble [19]. We also choose $D_x=0$ and $B_x=0$, taking into account the only polarization of electromagnetic waves. The simplified Maxwell equations therefore look as:

$$\begin{aligned}\partial_t D_y&=-\partial_x H_z\\ \partial_t B_z&=-\partial_x E_y\end{aligned} \tag{5}$$

Then, we omit indices for transparency of the formalism view. We introduce four fields $\mathcal{E}$, $\mathcal{B}$, $\mathcal{D}$, $\mathcal{H}$ as the Fourier images of E, B, D and H that are connected by inverse Fourier transformations:

$$\begin{aligned}E(x,t)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{E}(x,\omega)\exp(i\omega t)d\omega\\ B(x,t)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{B}(x,\omega)\exp(i\omega t)d\omega\\ D(x,t)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{D}(x,\omega)\exp(i\omega t)d\omega\\ H(x,t)&=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{H}(x,\omega)\exp(i\omega t)d\omega\end{aligned} \tag{6}$$

The domain of Fourier images is called a frequency domain or, alternatively, the ω-representation. The original functions $E(x,t)$, $B(x,t)$, $D(x,t)$, $H(x,t)$

are named as t-representation or a time domain. Linear material equations in the frequency domain is considered as originated from the quantum theory or experiment:

$$\mathcal{D} = \varepsilon_0 \varepsilon(\omega) \mathcal{E} \tag{7}$$

$$\mathcal{B} = \mu_0 \mu(\omega) \mathcal{H} \tag{8}$$

Here: $\varepsilon(\omega)$ – the dielectric permittivity of a medium in the frequency domain, ε_0 – the dielectric permittivity of the vacuum; $\mu(\omega)$ – the correspondent magnetic permeability of a medium and μ_0 – the magnetic permeability of the vacuum; $\mathcal{B}$ – the analogue of function B in ω-representation. For calculation purposes we need to use t-representation. In this representation ε and μ become integral operators of the convolution type:

$$\hat{\varepsilon}\psi(x,t) = \int_{-\infty}^{\infty} \tilde{\varepsilon}(t-s)\psi(x,s)ds$$

$$\hat{\varepsilon}^{-1}\psi(x,t) = \int_{-\infty}^{\infty} \tilde{e}(t-s)\psi(x,s)ds \tag{9}$$

$$\hat{\mu}\psi(x,t) = \int_{-\infty}^{\infty} \tilde{\mu}(t-s)\psi(x,s)ds$$

$$\hat{\mu}^{-1}\psi(x,t) = \int_{-\infty}^{\infty} \widetilde{m}(t-s)\mathcal{B}(x,s)ds \tag{10}$$

with kernels

$$\begin{aligned}
\tilde{\varepsilon}(t-s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon(\omega)\exp\Big(i\omega(t-s)\Big)d\omega \\
\tilde{e}(t-s) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \varepsilon^{-1}(\omega)\exp\Big(i\omega(t-s)\Big)d\omega \\
\tilde{\mu}(t-s) &= \frac{\mu_0}{2\pi}\int_{-\infty}^{\infty} \mu(\omega)\exp\Big(i\omega(t-s)\Big)d\omega \\
\widetilde{m}(t-s) &= \frac{1}{2\pi\mu_0}\int_{-\infty}^{\infty} \mu^{-1}(\omega)\exp\Big(i\omega(t-s)\Big)d\omega
\end{aligned} \tag{11}$$

Hence:

$$\begin{aligned}
D(x,t) &= \hat{\varepsilon}E(x,s)ds \quad & E(x,t) &= \hat{\varepsilon}^{-1}D(x,t) \\
B(x,t) &= \hat{\mu}H(x,t) \quad & H(x,t) &= \hat{\mu}^{-1}B(x,t)
\end{aligned} \tag{12}$$

The transforms define the fields and the material dispersion relation in the time domain, using the conventional continuation of the fields to the half space $t < 0$

and the causality condition [20]. To close the boundary regime problem, we add the boundary regime conditions:

$$E(0,t) = j(t) \quad B(0,t) = k(t) \tag{13}$$

prolonged to the $t \in (-\infty, 0)$ range in an appropriate way, that is adjusted to the experiment.

3. Projecting operators for the boundary regime problem

There is a symmetry with respect to the interchange of the independent variables x and t. The equations for the electromagnetic field components E and B (5) are written with use of the dispersion operators as (10), introduced in the previous sections:

$$\begin{gathered} \partial_t(\hat{\varepsilon}E) = -\partial_x(\hat{\mu}^{-1}B) \\ \partial_t B = -\partial_x E \end{gathered} \tag{14}$$

The action of operators $\hat{\varepsilon}$ and $\hat{\mu}$ was defined by (9)–(10).

Using the Fourier transformation leads to the system:

$$\begin{aligned} \partial_x \mathcal{B} &= -i\omega a^2(\omega)\mathcal{E} \\ \partial_x \mathcal{E} &= -i\omega \mathcal{B} \end{aligned} \tag{15}$$

here:

$$a^2(\omega) \equiv \mu_0 \varepsilon_0 \varepsilon(\omega)\mu(\omega) \equiv c^{-2}\varepsilon(\omega)\mu(\omega) \tag{16}$$

where c is the velocity of light in vacuum:

$$c^2 = \frac{1}{\varepsilon_0 \mu_0} \tag{17}$$

We write this system in matrix form:

$$\partial_x \widetilde{\Psi} = \mathcal{L}\widetilde{\Psi} \tag{18}$$

where matrices $\widetilde{\Psi}$ and $\mathcal{L}$ are:

$$\widetilde{\Psi} = \begin{pmatrix} \mathcal{B} \\ \mathcal{E} \end{pmatrix} \tag{19}$$

$$\mathcal{L} = \begin{pmatrix} 0 & -i\omega a^2(\omega) \\ -i\omega & 0 \end{pmatrix} \tag{20}$$

Equation (18) is a system of ordinary differential equations with constant coefficients that have exponential-type solutions. Following the technique described in [19], we arrive at a 2×2 eigenvalue problem. Let us look for such matrices $P^{(i)}$, $i = \overline{1,2}$ that $P^{(i)}\Psi = \Psi_i$ would be eigenvectors of the evolution matrix (20). The standard properties of orthogonal projecting operators:

$$\begin{gathered} P^{(1)}P^{(2)} = 0 \\ P^{(i)}P^{(i)} = P^{(i)} \\ P^{(1)} + P^{(2)} = I \end{gathered} \tag{21}$$

are implied. The inverse Fourier transformation $\widehat{\boldsymbol{P}}^{(i)} = \mathcal{F} P^{(i)} \mathcal{F}^{-1}$, where $\mathcal{F}$ – operator of Fourier transformation:

$$\mathcal{F}\Psi = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \int\limits_{-\infty}^{\infty} \mathcal{B}\exp(i\omega t)d\omega \\ \int\limits_{-\infty}^{\infty} \mathcal{E}\exp(i\omega t)d\omega \end{pmatrix} \tag{22}$$

leads to projectors in t-representation:

$$\widehat{\boldsymbol{P}}^{(1,2)}(t) = \frac{1}{2}\begin{pmatrix} 1 & \mp\hat{a} \\ \mp\hat{a}^{-1} & 1 \end{pmatrix} \tag{23}$$

where

$$\begin{aligned} \hat{a}\eta(x,t) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\eta(x,\tau)\int\limits_{-\infty}^{\infty} a(\omega)\exp\Big(i\omega(t-\tau)\Big)d\omega\right]d\tau \\ \hat{a}^{-1}\xi(x,t) &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\left[\xi(x,\tau)\int\limits_{-\infty}^{\infty} \frac{1}{a(\omega)}\exp\Big(i\omega(t-\tau)\Big)d\omega\right]d\tau \end{aligned} \tag{24}$$

4. Separated equations and definition of directed left and right waves

In t-representation, matrix equation (18) takes the form:

$$\partial_x\Psi = \widehat{L}\Psi \tag{25}$$

where

$$\Psi = \begin{pmatrix} B \\ E \end{pmatrix} \tag{26}$$

$$\widehat{L} = \begin{pmatrix} 0 & -\partial_t\widehat{a^2} \\ -\partial_t & 0 \end{pmatrix} \tag{27}$$

It can be checked that the operator $\widehat{a^2}$, defined as:

$$\widehat{a^2}\psi(x,t) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} a^2(\omega)\exp(i\omega t - i\omega\tau)\psi(x,\tau)d\omega d\tau \tag{28}$$

acts as a square of $\hat{a}$, defined by (24).

Making similar calculations, we can find, that $\hat{a}^2$ is expressed as the product $\hat{a}^2 = \hat{\varepsilon}\hat{\mu}$, that commutes

$$\hat{\varepsilon}\hat{\mu}\psi(x,t) = \hat{\mu}\hat{\varepsilon}\psi(x,t) \tag{29}$$

We note that this relation is true only if operators $\hat{\varepsilon}$ and $\hat{\mu}$ are convolution type integrals. For the further operations we also prove the commutation of operators ∂_t and $\hat{a}^2$.

Acting by the operator $\widehat{\boldsymbol{P}}^{(1)}$ (23) on the equation (25) we can commute $\widehat{\boldsymbol{P}}^{(1)}$ and ∂_x, because projectors do not depend on x. Using also the proven relations, we write

$$\partial_x \widehat{\boldsymbol{P}}^{(1)}(t)\Psi = \widehat{\boldsymbol{P}}^{(1)}(t)\widehat{L}\Psi = \widehat{L}\widehat{\boldsymbol{P}}^{(1)}(t)\Psi \tag{30}$$

After substituting Ψ and $\widehat{L}$ (26)–(27) and $\widehat{\boldsymbol{P}}^{(1)}$ (23) we find:

$$\partial_x \begin{pmatrix} \frac{1}{2}B + \frac{1}{2}\hat{a}E \\ \frac{1}{2}\hat{a}^{-1}B + \frac{1}{2}E \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}\hat{a}\partial_t B - \frac{1}{2}\hat{a}^2\partial_t E \\ -\frac{1}{2}\partial_t B - \frac{1}{2}\hat{a}^{-1}\hat{a}^2\partial_t E \end{pmatrix} \tag{31}$$

Applying the projection operators to the vector Ψ (26), we can introduce new variables Π and Λ as:

$$\Lambda \equiv \frac{1}{2}(B - \hat{a}E) \tag{32}$$

$$\Pi \equiv \frac{1}{2}(B + \hat{a}E) \tag{33}$$

Those are left and right (hybrid) waves variables. From (31) and a correspondent variable obtained by $\widehat{\boldsymbol{P}}^{(2)}$ we get two equations that determine evolution with respect to x of the boundary regime (13):

$$\begin{aligned} \partial_x \Pi(x,t) &= -\hat{a}\partial_t \Pi \\ \partial_x \Lambda(x,t) &= \hat{a}\partial_t \Lambda \end{aligned} \tag{34}$$

Using relations (33)–(32) from (13) we derive the boundary regime conditions for left and right waves:

$$\begin{aligned} \Lambda(0,t) &= \frac{1}{2}\Big(B(0,t) - \hat{a}E(0,t)\Big) = \frac{1}{2}\Big(k(t) - \hat{a}j(t)\Big) \\ \Pi(0,t) &= \frac{1}{2}\Big(B(0,t) + \hat{a}E(0,t)\Big) = \frac{1}{2}\Big(k(t) + \hat{a}j(t)\Big) \end{aligned} \tag{35}$$

This is a system of operator equations for a time-domain dispersion of left and right waves in a linear case.

5. A general system of interaction of nonlinear dispersive waves

Let us consider a nonlinear problem. We start again from the Maxwell's equations (5) with generalized material relations:

$$D = \hat{\varepsilon}E + P_{\mathsf{NL}} \quad B = \hat{\mu}H + M_{\mathsf{NL}} \tag{36}$$

where P_{NL} – the nonlinear part of polarization, M_{NL} – the part for magnetization. For our purposes linear parts of polarisation and magnetization have already been taken into account. In the time-domain, a closed nonlinear version of (14) is:

$$\begin{aligned} \partial_t(\hat{\varepsilon}E) + \partial_t P_{\mathsf{NL}} &= -\partial_x \hat{\mu}^{-1}B - \partial_x \hat{\mu}^{-1}M_{\mathsf{NL}} \\ \partial_t B &= -\partial_x E \end{aligned} \tag{37}$$

The operator $\hat{\mu}$ acting on the first equation of system (37) and using the same notations Ψ and $\widehat{L}$ from (26)–(27) once more, we obtain a nonlinear analogue of the matrix equation (25):

$$\partial_x \Psi - \widehat{L}\Psi = -\begin{pmatrix} \partial_x M_{\mathsf{NL}} \\ 0cr \end{pmatrix} - \begin{pmatrix} \hat{\mu}\partial_t P_{\mathsf{NL}} \\ 0 \end{pmatrix} \tag{38}$$

We introduce a vector of nonlinearity:

$$\mathbb{N}(E,B)=\begin{pmatrix}\partial_x M_{\mathsf{NL}}+\partial_t\hat{\mu}P_{\mathsf{NL}}\\ 0\end{pmatrix} \tag{39}$$

Then, we get a nonlinear analogue of the matrix equation (25):

$$\partial_x\Psi-\widehat{L}\Psi=-\mathbb{N}(E,B) \tag{40}$$

Next, acting by operators $\widehat{\boldsymbol{P}}^{(1)}$ (23) and $\widehat{\boldsymbol{P}}^{(2)}$ on Equation (40) we find:

$$\begin{aligned}\partial_x\Pi+\hat{a}\partial_t\Pi&=\mathbb{N}_1(\hat{a}^{-1}(\Pi-\Lambda),\Pi+\Lambda)\\ \partial_x\Lambda-\hat{a}\partial_t\Lambda&=-\mathbb{N}_1(\hat{a}^{-1}(\Pi-\Lambda),\Pi+\Lambda)\end{aligned} \tag{41}$$

where

$$\mathbb{N}_1(E,B)\equiv\frac{1}{2}(\partial_x M_{\mathsf{NL}}+\partial_t\hat{\mu}P_{\mathsf{NL}}) \tag{42}$$

Equation (42) is a system of equations of interaction of left and right waves due to arbitrary nonlinearity with a general temporal dispersion account. It is a principal result of this paper.

6. Equations of wave propagation in metamaterial described by lossless Drude model

6.1. Approximations for Drude model for dispersion

To obtain negative values of the constitutive parameters ε and μ, metamaterials must be dispersive, i.e., their permittivity and permeability must be frequency dependent, otherwise they would not be causal [21]. As it is shown in [22], if we have a frequency dispersion, the full energy density of the electromagnetic field will be:

$$W=\frac{d\big(\omega\varepsilon(\omega)\big)}{d\omega}E^2+\frac{d\big(\omega\mu(\omega)\big)}{d\omega}H^2 \tag{43}$$

$W>0$ if:

$$\frac{d\big(\omega\varepsilon(\omega)\big)}{d\omega}>0\quad\frac{d\big(\omega\mu(\omega)\big)}{d\omega}>0 \tag{44}$$

This does not contradict with simultaneously negative $\varepsilon<0$ and $\mu<0$ [4].

Materials with a typical plasma dispersion (the Drude formula of Lorentz origin) for both $\varepsilon(\omega)$ and $\mu(\omega)$ are often discussed. The Drude model is a limit case of the classical Lorentz model and represents a situation of a main contribution of free electrons that explains its use in the elementary conductivity theory, plasma physic and metamaterials. For this case we use the relations from [12]:

$$\varepsilon(\omega)=\left(1-\frac{\omega_{\text{pe}}^2}{\omega^2}\right)\quad\mu(\omega)=\left(1-\frac{\omega_{\text{pm}}^2}{\omega^2}\right) \tag{45}$$

This model is used by many authors, [23, 24] *et al.*, to describe the material properties of a metamaterial. The energy density (43) is positive at the ω range for which $\varepsilon(\omega)$ and $\mu(\omega)$ (45) are valid:

$$W=\left(1+\frac{\omega_{\text{pm}}^2}{\omega^2}\right)E^2+\left(\frac{\omega_{\text{pm}}^2}{\omega^2}+1\right)H^2>0 \tag{46}$$

where ω_{pe} and ω_{pm} – parameters, dependent on the density, charge, and mass of the charge carrier. These parameters are commonly known as the electric and magnetic plasma frequencies [12]. The kernel $a(\omega)$ of the operator $\hat{a}$ is:

$$a(\omega)=c^{-1}\sqrt{\left(1-\frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right)\left(1-\frac{\omega_{\mathrm{pm}}^2}{\omega^2}\right)} \tag{47}$$

After expansion $a(\omega)$ in the Taylor series in conditions of $\omega \ll \omega_{\mathrm{pe}},\omega_{\mathrm{pm}}$, in a vicinity of $\omega=0$, we get:

$$\hat{a}\eta(t)\approx c^{-1}\left[\omega_{\mathrm{pe}}\omega_{\mathrm{pm}}\partial_t^{-2}-\frac{1}{2}\frac{\omega_{\mathrm{pe}}^2+\omega_{\mathrm{pm}}^2}{\omega_{\mathrm{pe}}\omega_{\mathrm{pm}}}+\left(\frac{1}{2\omega_{\mathrm{pe}}\omega_{\mathrm{pm}}}+\frac{1}{8}\frac{(-\omega_{\mathrm{pe}}^2-\omega_{\mathrm{pm}}^2)^2}{\omega_{\mathrm{pe}}^3\omega_{\mathrm{pm}}^3}\right)\partial_t^2\right]\eta(t) \tag{48}$$

The operator ∂_α^{-1} is defined as the integral:

$$\partial_\alpha^{-1}f(\alpha)=\int\limits_0^\alpha f(\beta)d\beta \tag{49}$$

As it is seen from the numerical analysis (see Figure 1 a) in the range of frequencies $\omega < 0.5\omega_{\mathrm{pe}}$ the relative error of the expansion is less than 0.005%. For this frequency range the first term demonstrates the acceptable error of less than 25%. We can apply the mentioned case:

$$\hat{a}\eta(t)\approx c^{-1}\omega_{\mathrm{pe}}\omega_{\mathrm{pm}}\partial_t^{-2}\eta(t) \tag{50}$$

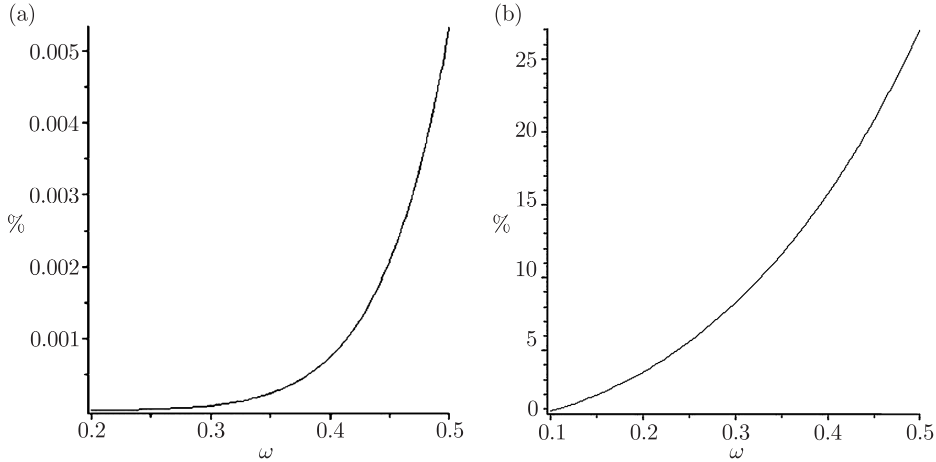

Figure 1. Relative error (in percent) of Taylor expansion (48) at range $[0.2, 0.5]$ ω_{pe} (a) and one for the first terms of (48) at range $[0.1, 0.5]$ ω_{pe} (b)

In the case of $\omega_{\mathrm{pm}}=\omega_{\mathrm{pe}}$ we find:

$$ca(\omega)=\sqrt{\left(1-\frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right)\left(1-\frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right)}=\sqrt{\left(1-\frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right)^2}=\left(1-\frac{\omega_{\mathrm{pe}}^2}{\omega^2}\right) \tag{51}$$

that already have algebraic form.

Taking into account all estimations, we leave the only term in the relation (48). Next, for compactness, we mark ω_{pe} as p, and ω_{pm} as q. Plugging this minimal version of (48) in the system (34) we obtain:

$$\partial_x \Pi = -c^{-1} pq \partial_t^{-1} \Pi \quad \partial_x \Lambda = c^{-1} pq \partial_t^{-1} \Lambda \tag{52}$$

Differentiating this system on t once more, we write the resulting system, in which the right and left wave amplitudes are completely separated

$$\partial_{xt} \Pi = -c^{-1} pq \Pi \quad \partial_{xt} \Lambda = c^{-1} pq \Lambda \tag{53}$$

Both equations describe the wave dispersion, they are equivalent to the 1+1 Klein-Gordon-Fock equation $\Box \phi_\pm = m_\pm \phi_\pm$ with the mass parameter $m_\pm = \pm c^{-1} pq$.

6.2. Kerr nonlinearity account for lossless Drude metamaterials; interaction of left and right waves

For nonlinear Kerr materials [25], the third-order nonlinear part of polarization [20, 19] has the form:

$$P_{\mathsf{NL}} = \chi^{(3)} E^3 \tag{54}$$

From (42) we find $\mathbb{N}_1$:

$$\mathbb{N}_1 \equiv \frac{1}{2} \hat{\mu} \left(\hat{\mu}^{-1} \partial_x M_{\mathsf{NL}} + \partial_t P_{\mathsf{NL}} \right) = \frac{\chi^{(3)}}{2} \hat{\mu} \partial_t E^3 \tag{55}$$

The operator $\hat{\mu}$ for the chosen model is just $\mu_0(1 - q^2 \partial_t^{-2})$. Further, the effect of negative permeability was demonstrated at the THz range [26]. Hence, the $q^2 \partial_t^{-2}$ contribution prevails. Then, from (41) we obtain:

$$\begin{aligned} c\partial_x \Pi - pq \partial_t^{-1} \Pi &= -\frac{\chi^{(3)}}{2} \mu_0 q^2 \partial_t^{-1} \left[\hat{a}^{-1} (\Pi - \Lambda) \right]^3 \\ c\partial_x \Lambda + pq \partial_t^{-1} \Lambda &= \frac{\chi^{(3)}}{2} \mu_0 q^2 \partial_t^{-1} \left[\hat{a}^{-1} (\Pi - \Lambda) \right]^3 \end{aligned} \tag{56}$$

The same approximation for the operator $\hat{a}^{-1}$ reads as:

$$\hat{a}^{-1} \eta(x,t) \approx \frac{c}{pq} \partial_t^2 \eta(x,t) \tag{57}$$

We substitute it to the system (56) and differentiate it, denoting derivatives by indices for more compactness:

$$\begin{aligned} c\Pi_{xt} + pq\Pi &= -\frac{\mu_0 \chi^{(3)} c^3}{2p^3 q} \left[(\Pi - \Lambda)_{tt} \right]^3 \\ c\Lambda_{xt} - pq\Lambda &= \frac{\mu_0 \chi^{(3)} c^3}{2p^3 q} \left[(\Pi - \Lambda)_{tt} \right]^3 \end{aligned} \tag{58}$$

We consider this system as **the main result of our work**. The equivalent system is obtained by triple differentiation of both equations of the system with respect to time and rescaling $\Pi_{tt} = \alpha \pi$, $\Lambda_{tt} = \alpha \lambda$, $x = \beta \zeta$ with the choice $\alpha = \sqrt{\frac{2p^4 q^2}{\mu_0 \chi^3 c^3}}$, $\beta = \frac{c}{pq}$. Then

$$\begin{aligned} \pi_{\zeta t} + \pi &= -\left[(\pi - \lambda)^3 \right]_{tt} \\ \lambda_{\zeta t} - \lambda &= \left[(\pi - \lambda)^3 \right]_{tt} \end{aligned} \tag{59}$$

with extra boundary conditions.

Consider the unidirectional case of (58) with $\Lambda = 0$, that corresponds to special initial conditions from (35): $\Big(k(t) - \hat{a}j(t)\Big) = 0$ and is valid till the effect of the left wave generation is noticeable.

$$c\Pi_{xt} + pq\Pi = -\frac{\mu_0 \chi^{(3)} c^3}{2p^3 q} [\Pi_{tt}]^3 \tag{60}$$

7. Stationary solution in a moving reference frame

7.1. Linear case

We introduce a change of variables

$$x = \eta \quad \xi = x - vt \tag{61}$$

v has the dimension of speed. We declare the independence of R and L from η as a definition of the stationary state:

$$\partial_\eta \mathsf{R} = \partial_\eta \mathsf{L} = 0 \tag{62}$$

$$\begin{aligned} -v\partial_\xi^2 \mathsf{R} &= -pqc^{-1}\mathsf{R} \\ -v\partial_\xi^2 \mathsf{L} &= pqc^{-1}\mathsf{L} \end{aligned} \tag{63}$$

The dimension of r.h.s. is a dimension of $\mathsf{k}\omega$:

$$\mathsf{k}\omega = \frac{pq}{c} \tag{64}$$

Also we find v:

$$v = \frac{\omega}{\mathsf{k}}. \tag{65}$$

We start with the R wave only. We rewrite it taking into account (64)–(65):

$$\partial_\xi^2 \mathsf{R} - \mathsf{k}^2 \mathsf{R} = 0 \tag{66}$$

We solve the boundary problem by means of the solution domain specified by $x > 0$, $t > 0$. For a decaying boundary regime for $v > 0$ the solution is:

$$\mathsf{R} = A \exp(\mathsf{k}(x - vt)) \tag{67}$$

For the L-wave the equation differs only by a sign from (66):

$$\partial_\xi^2 \mathsf{L} + \mathsf{k}^2 \mathsf{L} = 0 \tag{68}$$

that gives the oscillating solution:

$$\mathsf{L} = B \sin\Big(\mathsf{k}(x - vt)\Big) \tag{69}$$

As we can see, the negative value for μ drastically changes the character of propagation of the waves R and L, the definition of which is given by (32)–(33).

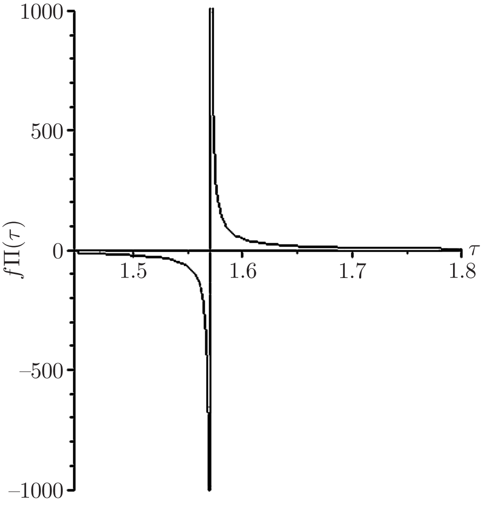

Figure 2. Solution of nonlinear equation (70) for auxiliary function $f\Pi(\tau)$, where $f^2 = \frac{\mu_0 \chi^{(3)}}{\mathsf{k} v}$, $\xi = \mathsf{k}\tau$, for $p = 10^9$ Hz, $q = 0.8p$, $\omega = 0.5p$

7.2. On nonlinear case

Equation (60) after transition to variables (61) and the use of stationary condition is as follows:

$$c\Pi_{\xi\xi} + pq\Pi = -\frac{\mu_0 \chi^{(3)} c^3 v^6}{2p^3 q} \left[\Pi_{\xi\xi}\right]^3 \tag{70}$$

Solving the cubic equation with respect to $\Pi_{\xi\xi}$ by the Cardano formula, one arrives at a rather complicated nonlinear oscillator

$$\Pi_{\xi\xi} = F(\Pi) \tag{71}$$

the expansion of which with respect to Π up to Π^3 term yields equations that may be directly integrated in terms of elliptic functions [27]. The limit case of the singular solitary wave solution is presented in Figure 2.

8. Conclusion

Using the projection operator approach we have derived a general system of equations (41) that describes the interaction between opposite directed waves propagating in a 1D-metamaterial with an arbitrary dispersion and nonlinearity. The system is specified for a lossless Drude dispersion and the Kerr nonlinearity model as (58). A stationary solution derivation is outlined within the weak nonlinearity approximation.

The manipulations by means of the projecting technique are a part of the symbolic-numerical computation program. The resulting equations are prepared for an effective numerical solution because the direction of finite difference integration is determined by the definition of new (left/right) dynamical variables. An investigation and a numerical solution of the obtained system of equations are planned to be published in the nearest future. There is a promising approach to the SW equation by the so-called polysimplectic integration in [28]

The results may be used in experiments that investigate the amplitude dependence of the reflected wave on a metamaterial layer.

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