PROBLEM OF DISTURBANCE IDENTIFICATION BY MEASUREMENT IN THE VICINITY OF A POINT

SERGEY LEBLE[1,2] AND IRINA VERESHCHAGINA[2]

[1] *Faculty of Applied Physics and Mathematics*
Gdansk University of Technology
Narutowicza 11/12, 80-233 Gdansk, Poland

[2] *Department of Physics, Institute of Physics and Technology*
I. Kant Baltic Federal University
A. Nevskogo, 14, 236041, Kaliningrad, Russia

(received: 11 January 2016; revised: 19 February 2016;
accepted: 25 February 2016; published online: 30 March 2016)

Abstract: A problem of wave identification is formulated. We propose a diagnostic analysis of medium disturbances based on distinguishing of components of a wave vector that is specific for each kind of the wave mode. Mathematically it is realized by projection operator technique. An example is considered in conditions of a one-dimensional Cauchy problem for a conventional wave equation in the matrix form and its version with weakly x-dependent coefficients as a demonstration of the method application for the simplest adiabatic theory of one-dimensional acoustics. The case of acoustics in a gas with a dissipation account is also discussed from the point of view of the wave and entropy mode diagnostics.

Keywords: wave disturbances, acoustics

1. Introduction

The problem of wave mode identification is of an interdisciplinary nature, it is important, for example in physics of atmosphere, where superposition of acoustic, gravity and planetary waves occurs [1]. In the planetary range the periods of Rossby and Poincare waves are of the same scale, hence their separation and estimation of contributions is complicated. The situation is even more complicated in plasma physics where additional specific branches of waves coexist with ones for neutral gas. An important problem of the specific mode source localization is also a typical inverse problem that is generally ill-posed [2, 3].

In geophysics the wave field diagnostics generally needs many observations that cover the space sufficient for wave length estimation. It is rather expensive and not very feasible. Thinking about the novel alternative approach [4] we suggest to use measurements restricted by the vicinity of a point which however need many-component observations with the aid of the projection operator technique built in this article, fitting the subspaces of specific waves.

We would start with an instructive example of a 1D wave equation and a correspondent Cauchy problem (Section 2), see *e.g.* [5, 6]. Naturally dispersion or dissipation complicate the situation but may be overcome [7, 8]. Weak inhomogeneities of the propagation media may be also effectively included a in similar manner [9].

There are lot of important problems of theoretical physics with the same level of description: for electrodynamics, see [10–12] for acoustic [13, 14], and Tollmienn-Schlichting waves [15]. that may be directly formulated as a system of equations, so the vector description (in electrodynamics it is (E,B)) has a direct physical sense. It should be nonetheless mentioned that directed waves correspond to the so-called hybrid variables with appropriate initial or boundary conditions.

More complicated 3D problems need more advanced construction and the geometry (ring or sphere in geophysics [4] – *e.g.*) impact may lead to very non-trivial generalization of the technique and algorithm as well as a norm for appropriate construction of spaces.

Our present study is focused on the simplest $1+1$ case that includes one space and one time coordinate. We restrict ourselves mainly by a uniform medium, nevertheless, we would like to touch the base of the projection operator technique and the estimation of the quality of diagnostics in a finite number of measurements in terms of a physically reasonable Banach space. Hence, we could estimate the diagnostics errors and, therefore, the quality of the position of the wave source estimation. We focus our attention on the minimal version of the theory to show that the main idea of wave diagnostics may be most characteristic for the opposite propagating waves. In this $1+1$ case the wave type (polarization) is linked to the direction of propagation that allows us to formulate the whole algorithm of some inverse problem solution in the following form:

1) Reformulate a problem that fixes eigen subspaces of an evolution operator;
2) Project the subspaces and their weight evaluation in an appropriate physical norm for functional and finite-dimensional spaces;
3) Estimate the time arrival and wave form for a given number of measurements and choose the spline order;
4) Estimate the distance to the area of initialization within the prescribed error limits;
5) Investigate the stability in terms of explicit solution form, reconstructed by the finite points number data. It should be mentioned that the last task relates to the analytical continuation problem [2, 3].

Such an algorithm is realized in Section 2 for the Cauchy problem for the homogeneous string equation example and developed in Section 3 for a more general system and weak inhomogeneity account. A development of the theory with dissipation and entropy mode account in Section 4.

2. String equation in vector form. Cauchy problem in terms of projectors

The conventional Cauchy problem for the wave equation contains two initial conditions, including the time derivative, needs measurement of the derivative, hence its physical version includes several points in the vicinity of the point of observation and estimation of the diagnostics error in an appropriate space. The conventional Cauchy problem for the $1+1$ wave equation

$$u_{tt} - c^2 u_{xx} = 0 \tag{1}$$

with the initial data

$$u(x,0) = \phi(x) \qquad v(x,0) = \psi(x) \tag{2}$$

has the matrix representation in terms of a vector

$$\psi^T = (cu_x = v, u_t = w) \qquad D = \frac{\partial}{\partial x} \tag{3}$$

for the components (3) reads

$$\begin{aligned} v_t - cw_x &= 0 \\ w_t - cv_x &= 0 \end{aligned} \tag{4}$$

and has the evolution matrix operator L that appears in the system

$$\psi_t = cD\begin{pmatrix} w \\ v \end{pmatrix} = cD\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\psi = L\psi \tag{5}$$

It is convenient to solve Equation (5) by means of the projection operator [7, 8] technique. The projectors for the system (4) (see again [7, 8]) are almost obvious

$$P_\pm = \frac{1}{2}\begin{pmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{pmatrix}. \tag{6}$$

The identity

$$(P_+ + P_-)\psi = \psi \tag{7}$$

in terms of

$$P_+\psi = \begin{pmatrix} \Pi \\ \Pi \end{pmatrix} \tag{8}$$

and

$$P_-\psi = \begin{pmatrix} \Lambda \\ -\Lambda \end{pmatrix} \tag{9}$$

reads

$$\begin{aligned} \Pi &= \frac{1}{2}(v+w) \\ \Lambda &= \frac{1}{2}(v-w) \end{aligned} \tag{10}$$

For details and generalization see [9].

Even in this simplest case the physical content is nontrivial because the right (left) wave is of a hybrid form (see [7, 8, 10]), recall the definition of the variables v, w constant derivatives (3), therefore $\Pi = \frac{1}{2}(cu_x + u_t)$. As it is known, the derivative evaluation in a finite-difference context is ill-posed and needs regularization [2, 3].

The projection separates the space Ψ into a direct sum of subspaces

$$\Psi = \Psi_+ \oplus \Psi_- \qquad \begin{pmatrix} \Pi \\ \Pi \end{pmatrix} \in \Psi_+ \qquad \begin{pmatrix} \Lambda \\ -\Lambda \end{pmatrix} \in \Psi_- \tag{11}$$

Applying the projectors directly to (5) yields

$$(P_\pm \psi)_t = cDP_\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi \tag{12}$$

or

$$\begin{aligned} \Pi_t + c\Pi_x &= 0 \\ \Lambda_t - c\Lambda_x &= 0 \end{aligned} \tag{13}$$

The technique is convenient for analysis of a general problem of diagnostics and its quality estimation because the solution space splits into spaces with simpler evolution. For a one-dimensional string case it is right and left waves only that are taken into account.

Let a right wave Π arrive at a point of observation, say $x = 0$, giving in measurement the vector ψ components value in a time sequence. The action of the „left" projector P_- to such a vector gives zero, if function Π is an exact solution of the Equation (13).

Suppose the vector space of solutions is attributed by a norm, $\psi \in B$, hence B is a Banach space. In a general diagnostic exposition introduce a normalized solutions λ, π of more complicated situation when the disturbance of the string $\alpha\pi + \beta\lambda$ may arrive at $x = 0$ from both sides simultaneously, then the action of the projectors $P_\pm$ cuts one of the waves with the result, for example

$$P_+ \psi = P_+ \left[\alpha \begin{pmatrix} \pi \\ \pi \end{pmatrix} + \beta \begin{pmatrix} \lambda \\ -\lambda \end{pmatrix} \right] = \alpha \begin{pmatrix} \pi \\ \pi \end{pmatrix} \tag{14}$$

Unfortunately real measurement gives information about ψ components with errors and only in a finite number of time points. So we should estimate the distance between the representative observation and the space of, say, „exact" right waves.

Let us introduce a norm in the vector space of solutions, decaying at infinities exponentially. It is directly verified by (13)

$$\left[\int_{-\infty}^{\infty} \left(\Lambda^2 + \Pi^2 \right) dx \right]_t = 0 \tag{15}$$

therefore it is convenient to introduce the norm via this conservation law (15)

$$\|\psi\|^2 = \int_{-\infty}^{\infty} \left(\Lambda^2 + \Pi^2 \right) dx \tag{16}$$

or in terms of original components

$$\|\psi\|^2 = \int_{-\infty}^{\infty} \left(\frac{1}{2} \left(v^2 + w^2 \right) \right) dx, \tag{17}$$

because $\Lambda^2+\Pi^2=\frac{1}{4}\left[(v+w)^2+(v-w)^2\right]$ The integral is proportional to energy of a string.

In such a way a Banach space Ψ is introduced to estimate the distances in the space of $\psi\in\Psi$, with normalized $\|\lambda\|=\|\pi\|=1$.

Given a sequence of times t_i, $i=1,\dots n$ generates a set of vectors $\phi(0,t_i)$, by measurements, which form the 2n-dimensional vector $\phi\in R^{2n}$ with a norm

$$\|\phi\|_n^2=\sum_{i=1}^{n}\left(\phi_1^2(0,t_i)+\phi_2^2(0,t_i)\right) \tag{18}$$

that should determine a closest vector solution $\psi_+(x-ct)$ via the functional minimum

$$I=\left\|\psi_+-P_+\phi(0,t_i)\right\|_n \tag{19}$$

where $\psi_{i+}=\psi_+(0-ct_i)$. We treat this condition as variational principle

$$\min_{\psi_+\in\Psi_+} I \tag{20}$$

in a 2n-dimensional space with the norm (18) correspondent to L_2 (16) with the Euler equations

$$\frac{\partial I}{\partial\psi_{i+}}=0 \tag{21}$$

Let us recall (see (3)) that the vector of observation data has the components $\phi_1=c\frac{\Delta u}{\Delta x}\approx cu_x$, $c\frac{\Delta u}{\Delta t}\approx cu_t)$ that are obtained by measurements of the string function $u(x,t)$ in adjacent points of time and in $x=0$ and in $x=\Delta x$, that belong to the point $x=0$ vicinity. For example, if components in a sequence coincide $\left(\phi_1(t_i)=\phi_2(t_i)\right)$ the action of the left projector P_-

$$P_-\phi=\frac{1}{2}\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_1\end{pmatrix} \tag{22}$$

gives zero. In reality the errors of measurements do not give zero identically even if the wave is purely right. The deviation of the real data from the ideal may be characterized by

$$\left\|\phi_+-P_+\phi(0,t_i)\right\|_n=\delta \tag{23}$$

the correspondent calibration may be performed by a specially organized experiment.

An admixture of a left wave in a superposition of the both may be noticed if $\|\phi-P_+\phi\|_n$ exceeds the typical error. If noticed, the left wave reconstruction of the second step is made by the second projector P_-.

The final form of the reconstruction is a standard procedure of a spline or another inverse problem in a physically reasonable space [2, 3] that is chosen a priori. Note, that the arrival time needs a priori information about the „zero” time event.

An instability from velocity value errors should also be taken into account, error from the arrival time grows with it.

3. General hyperbolic problem. Inhomogeneity account

Here we admit an inhomogeneity of the medium of propagation in a more natural frame of the system for a couple of directly measured variables u,v. In acoustics it would be hydrodynamic velocity and pressure [13], in electromagnetism – electric and magnetic fields [16]. The tsunami problem is studied in [5] within a similar framework. Such a system and initial problem data for a vector $\psi^T = (u,v)$ do not contain a derivative in space, so their diagnostics needs time sequence measurements of both the vector components at the only point.

Consider the initial problem for the system

$$\frac{\partial u(x,t)}{\partial t} - \epsilon b(x)\frac{\partial v(x,t)}{\partial x} = 0, \tag{24}$$

$$\frac{\partial v(x,t)}{\partial t} - \epsilon c(x)\frac{\partial u(x,t)}{\partial x} = 0. \tag{25}$$

for $u(x,t),\ v(x,t) \in C^1,\ x \in (-\infty,\infty),\ t \geq 0,\ u(x,0) = \phi(x),\ v(x,0) = \psi(x)$. We introduce small parameter ϵ, to characterize the initial conditions

$$\max \frac{\partial \phi(x)}{\partial x} = \epsilon \ll 1 \qquad \max \frac{\partial \psi(x)}{\partial x} = \epsilon \ll 1 \tag{26}$$

The (weak) inhomogeneity is described by the dependence of system coefficients c, d on x. The dependence on the small parameter ϵ is implied and skipped in this text (see details in [9]).

The problem is reformulated in terms of directed waves. The projection operators in this case are calculated via the basic relation for the projection subspaces that are derived directly from Equations (24)–(25), in which evolution is fixed by the pseudodifferential spectral operators as expansion in ϵ [9]). In the first order, arriving at a supermatrix:

$$P_{1,2} = \frac{1}{2}\begin{pmatrix} 1 & \pm M^{-1} \\ \pm M & 1 \end{pmatrix} \tag{27}$$

where $f = \sqrt{\frac{c}{b}}$, $D = \frac{\partial}{\partial x}$ and the operator valued matrix elements are expressed in terms of $M = D^{-1} f D$.

Now the evolution operator L, in the same notations, is also a supermatrix:

$$L = \begin{pmatrix} 0 & \epsilon b(x) D \\ \epsilon c(x) D & 0 \end{pmatrix} \tag{28}$$

To proceed in the theory we base on the commutation relation $[P_{1,2}, L] = 0$ that is valid automatically in the case of constant coefficients b, c. For the x-dependent case of $b(x)$, $c(x)$ the commutator L and P_1 is equal to

$$[P_1, L] = \frac{\epsilon}{2}\begin{pmatrix} M^{-1}cD - bDM & 0 \\ 0 & MbD - cDM^{-1} \end{pmatrix} \tag{29}$$

The condition that the commutator is zero can be written as

$$D^{-1} f' b f = 0 \tag{30}$$

or plugging the expression for f gives:

$$c'b - b'c = 0 \tag{31}$$

It fixes the case of complete reduction (diagonalization) of the evolution operator.

As further development of the method we suggest an approximate procedure (see *e.g.* [15]), generally treating the condition $[P_{1,2}, L] = O(\epsilon)$.

Using the projection operators we reduce (24) to a couple of equations that, in the previous section were traditionally named as equations for left and right waves, splitting the problem of evolution. The approximate splitting is achieved if the commutators of $P_{1,2}$ and L can be neglected. It is possible, if coefficients b, c are of the zero order ($\cong O(1)$), while the order of the derivative $\left(\frac{c}{b}\right)'$ is of a higher order, *e.g.* $\cong O(\epsilon)$. It is guaranteed by the evolution operator dependence on ϵ and conditions of the spectral operators expansion domain. Acting by $P_{1,2}$ to the system (24)–(25)

$$(P_{1,2}\Psi)_t = P_{1,2}L\Psi \tag{32}$$

or, approximately

$$(P_{1,2}\Psi)_t = L(P_{1,2}\Psi) \tag{33}$$

where

$$P_1\Psi = \frac{1}{2}\begin{pmatrix} \Pi \\ M\Pi \end{pmatrix} \tag{34}$$

$$P_2\Psi = \frac{1}{2}\begin{pmatrix} \Lambda \\ -M\Lambda \end{pmatrix} \tag{35}$$

Reading the first lines of the relations yields

$$\Pi = \frac{1}{2}\left(u + M^{-1}v\right) \tag{36}$$

and

$$\Lambda = \frac{1}{2}\left(u - M^{-1}v\right) \tag{37}$$

This relation allows us to establish the Cauchy problems for directed waves.

$$\Pi(x,0) = \frac{1}{2}\left(\phi + M^{-1}\psi\right) \tag{38}$$

and

$$\Lambda(x,0) = \frac{1}{2}\left(\phi - M^{-1}\psi\right) \tag{39}$$

From the equations (36)–(39) one extracts:

$$\begin{aligned} u &= \Pi + \Lambda \\ v &= M(\Pi - \Lambda) \end{aligned} \tag{40}$$

Considering equations(28)–(32) and the relation for the commutator $P_1L = LP_1 - [P_1, L]$ one obtains approximately:

$$\Pi_t = -\sqrt{bc}\,\Pi_x \tag{41}$$

That could be interpreted as the equation for the right wave. Similarly the equation for the left wave variable Λ looks as follows

$$\Lambda_t = \sqrt{bc}\,\Lambda_x \tag{42}$$

Solving the first order equations by a method of characteristics gives u, v by the relation (40); formally the system coincides with (13) but the velocity of propagation and coefficients in (40) are functions depending on a coordinate.

The problem of diagnostics and reconstruction is formulated within the scheme of the previous section, it is based on the algorithm described in the introduction. The norm definition, as prescribed at the Banach space Ψ is based on left/right waves variables representation (16) may be reformulated from the conservation law

$$\left[\int_{-\infty}^{\infty}(b^{-1}u^2+c^{-1}v^2))dx\right]_t=0 \tag{43}$$

for an integrable integrand ($e.g.$ $b>0$, $c>0$). The correspondent functional is again invariant with respect to time shift. The functional (19) has the same form but the projector in it is modified as in (27). A perturbation, again settled at time $t=0$ as localized perturbation, propagates along the characteristics for the given inhomogeneities of the propagation medium.

We would note that the diagnostics may be performed using the approximate derivatives directly from equations (41)–(42) on each step of data including.

4. Acoustics. Entropy mode and dissipation account. Projection technique development

In this section we develop the results of [9] in the direction of [13] for a 1D homogeneous gas medium.

Basic systems for a viscous and thermo-conductive liquid are defined by the momentum, energy and mass balance. It can be written in dimensionless variables based on a characteristic length of the disturbance (λ), the linear speed of sound and density, (c_0 and ρ_0) such that the dimensionless value of the uniform background density is equal to 1. It can also be viewed in matrix form as

$$\frac{\partial}{\partial t}\psi+L\psi=\widetilde{\psi} \tag{44}$$

where L is the linear matrix operator

$$L=\begin{pmatrix} -\delta_1 D & 1 & 0 \\ 1 & -\frac{\gamma\delta_2}{\gamma-1}D & \frac{\delta_2}{\gamma-1}D \\ 1 & 0 & 0 \end{pmatrix} D \tag{45}$$

with $\delta_1=\frac{4\mu}{3\rho_0 c_0\lambda}$, and $\delta_2=\frac{\kappa}{\rho_0 c_0\lambda}\left(\frac{1}{c_v}-\frac{1}{c_p}\right)$. Here λ denotes a characteristic scale perturbation along the x variable. The constant parameters of the fluid μ, κ are viscosity and thermal conductivity, respectively. The heat capacities c_p, c_v are normalized per unit mass, $\gamma=c_p/c_v$. Here the state vector is defined as $\psi(x,t)=\begin{pmatrix} v(x,t) & p(x,t) & \rho(x,t)\end{pmatrix}^T$, where v represents the x-component of the non-dimensional unsteady velocity and p and ρ are non-dimensional perturbations of the pressure and density.

The formation of these projection operators for the given one-dimensional flow system gives

$$P_{1,2} = \begin{pmatrix} \frac{1}{2} \pm \left(\frac{\delta_2}{2} - \frac{\beta}{4}\right) D & \pm\frac{1}{2} - \frac{\delta_2}{2(\gamma-1)} D & \frac{\delta_2}{2(\gamma-1)} D \\ \pm\frac{1}{2} & \frac{1}{2} \pm \left(\frac{\beta}{4} - \frac{\gamma\delta_2}{2(\gamma-1)}\right) D & \pm\frac{\delta_2}{2(\gamma-1)} D \\ \pm\frac{1}{2} + \frac{\delta_2}{2} D & \frac{1}{2} \pm \left(\frac{\beta}{4} - \frac{\delta_2}{2(\gamma-1)}\right) D & \pm\frac{\delta_2}{2(\gamma-1)} D \end{pmatrix} \tag{46}$$

defining the left and right waves Π, Λ. While the third (approximate) projector

$$P_3 = \begin{pmatrix} 0 & \frac{\delta_2}{\gamma-1} D & -\frac{\delta_2}{\gamma-1} D \\ 0 & 0 & 0 \\ -\delta_2 D & -1 & 1 \end{pmatrix} \tag{47}$$

yields the so-called entropy mode s.

In an absorbing fluid, the total energy conservation law should be considered, the governing equation of interest is thus [13, 17]

$$(\rho e + \rho(\mathbf{v}\cdot\mathbf{v})/2)_t + \nabla\cdot\mathbf{J} = 0 \tag{48}$$

where $\mathbf{J} = p\mathbf{v} + e\mathbf{v}$ is the energy flux density vector, e, ρ, $\mathbf{v}$ and p are internal energy per mass unit, mass density, velocity, and pressure, respectively. For ideal gas $e = p/\rho(\gamma-1)$.

The division of the total perturbation field in accordance with the mode content and its substitution into (48) results in [13]

$$\frac{\partial E_s}{\partial t} = -\operatorname{div}\vec{J}_a \tag{49}$$

where E_s – the energy density of the entropy mode, J_a – the density of the energy flux for the acoustic waves. Both these are the results of averaging by period. So, we should take into account the energy losses when diagnostics is performed.

In the one-dimensional case the equation (48) reads

$$\left(\rho e + \rho v^2/2\right)_t + DJ = 0 \tag{50}$$

where $J = pv + ev$.

The action of operators splits the system of equations similarly as in previous sections. The norm and the whole analysis of measurements is more complicated because of the dissipation originally presented in the problem formulation. Formally, due to the mode evolution equations [13] for Π, Λ, s, the equality

$$\left[\int_{-\infty}^{\infty} (E_a + E_s)\,dx\right]_t = 0 \tag{51}$$

holds on rapidly decaying functions (localized perturbation), but the integrand is the energy perturbation density including the entropy part (for an explicit expression of E_a, E_s, see [18]). A measurement of the entropy part may be not available, then the balance (49) should be taken into account.

The energy losses of an acoustic wave are proportional to the dissipation and nonlinearity product, it is weak for long period waves and without very large amplitudes. Hence, the choice of norm may account only for the acoustic part,

$$\|\psi\|^2 = \int_{-\infty}^{\infty} E_a dx \tag{52}$$

that may be used but with its non-conservation account. The resulting algorithm of diagnostics in this case depends on the solution form and is similar to the one described in Section 1.

5. Conclusion

We have presented a 1+1 version of the theory of wave diagnostics. As it is demonstrated the key tools are non-trivially generalized, but the basic Banach space and the functional (19) are lifted in the prescribed form. The account of the weak inhomogeneity and dissipation is incorporated by modification of correspondent projection operators, going to a non-Abelian algebraic approach or including an extra (entropy) mode in the second case. The conservation law modification is necessary to introduce the appropriate norm for an estimation of the wave mode contribution.

6. Acknowledgements

The work is supported by the Ministry of Education and Science of the Russian Federation (Contracts No N GZ 3.1127.2014K).

References

[1] Pedlosky J 2003 *Waves in the Ocean and Atmosphere: Introdaction to Wave Dynamics*, Springer-Verlag

[2] Lavrent'ev M M 1967 *Some Improperly Posed Problems in Mathematical Physics*, Springer

[3] Lavrent'ev M M and Savel'ev L Ya 1995 *Linear Operators and Ill-Posed Problems*, Consultants Bureau, Division of Plenum Publishing Corporation

[4] Karpov I V, Bessarab F S and Leble S 2007 *Vestnik BFU im. I. Kanta, a series of Sci. Science* **4** 76 (in Russian)

[5] Romanov V G and Moshkalev P S 2011 *Sibirskii Zhurnal Industrial'noi Matematiki*, mathnet.ru

[6] Lavrent'ev M M, Reznitskay K G and Yakhno V G 1986 *AMS Trans. Ser. 2* **130**

[7] Leble S 1988 *Propagation of Nonlinear Waves in Stratified Media*, Leningrad University Press (in Russian)

[8] 1991 *Nonlinear Waves in Waveguides*, Springer-Verlag

[9] Leble S and Vereshchagina I 2014, http://arxiv.org/abs/1403.7751

[10] Kinsler P 2010 *Phys. Rev. A* **81**, 023808

[11] Kinsler P 2010 *Phys. Rev. A* **81**, 013819

[12] Kuszner M and Leble S *2011*, *J. Phys. Soc. Jpn.* **80**, 024002

[13] Perelomova A 2006 *Physics Letters A* **357** 42

[14] Perelomova A 2009 *Archives of Acoustics* **34** (2) 127

[15] Perelomova A and Leble S 2005 *TMF* **144** 1030

[16] Kinsler S B P and Radnor G H C 2005 *Phys. Rev. A* **72**, 063807
[17] Pierce A D 2002 *Nonlinear Acoustics at the Beginning of the 21st Century*, Rudenko O V and Sapozhnikov O A Eds., Faculty of Physics, MSU, **1** 11
[18] Makarov S and Ochmann M 1997 *Acta Acustica united with Acustica* **83** (2) 197