

GENERAL REMARKS ON DYNAMIC PROJECTION METHOD

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(received: 4 January 2016; revised: 12 February 2016;
accepted: 22 February 2016; published online: 21 March 2016)

Abstract: A brief history and a mathematical description of the dynamic projection operators technique is presented. An example of the general Cauchy problem for evolution equations in $1+1$ dimensions is studied in detail. A boundary regime propagation is formulated in terms of operators and illustrated by the simplest one-dimensional diffusion equation. The problem of temperature waves is discussed.

Keywords: partial differential equations, dynamical system, Cauchy problem, boundary regime propagation, diffusion equation, projection operator, idempotent operators, eigen mode, mode interaction

1. Introduction. A brief history

Numerical evaluation of a model parameter is an effective tool to present the results of modeling in form of tables or plots. Such representation of results is a necessary component of theoretical physics [1] and is effectively used in the range from nuclear to universe scales. In many cases there are difficulties related to such direct modeling, if different scales enter the simulated phenomenon. A good example of the case is a situation in which waves of drastically different frequencies are excited. It happens when the basic system of equations is multicomponent. In such situation it is effective to separate the description into parts with the different characteristic scales. Mathematically we split the solution space into correspondent subspaces to follow the calculations with the necessary steps in the underlying coordinates. The solution space can be divided by means of projection operators, built on eigenvectors of the evolution operator of the problem under consideration.

Hence, we suggest a combined analytic – numerical scheme of integration of a multicomponent system of partial differential equations with two kinds of

initial-boundary conditions. The problem we are going to solve is either of the initial (Cauchy) or boundary regime propagations.

The main precursors of the projection method are to be found in the papers of Boa-Teh Chu, L. S. G. Kovasznay [2]. The authors proposed a division of the space of an evolution equation of linear dissipative hydrodynamics into subspaces related to the roots of the dispersion equation and correspondent links between components of a state vector. The wave vector and, therefore – the frequency, are introduced via the Fourier transformation in space coordinates, which is effective in the case of a homogeneous basic state (coefficients of the equations are independent from the coordinates). Links between dynamical variables are specified by the eigenvectors of the representation of the evolution operator for such subspaces, parameterized by the wavevector. Hence, such link defines a fixed combination of original field components after the inverse Fourier transform. Let us also mention an introduction of the so-called combined or coupled waves (amplitudes) in electronics [3], natural forms [4] and “travelling wave” used in the laser theory [5].

The starting point for our group in a direction of this method was connected with nonlinear generalization of the projection method with the first touch in the diploma work of I. Vereshchagina under my supervision [6]. Its explicit form in terms of the corresponding projection matrix operators in k -space for the Cauchy problem of acoustic and internal waves in the $2+1$ exponential atmosphere without dissipation were published in the Khabarovsk conference paper [7] and in a bit advanced form in the Novosibirsk international IUPAP-IUTAM conference paper [8]. In these papers the nonlinear generalization is given for long waves, already in x -representation, it results in a Zakharov-like system describing the interaction of internal and acoustic waves. Some details and development are presented in the book [9], where a useful formula for the matrix elements of the idempotents has been written. Further generalization for general hydrodynamics, including planetary (Rossby and Poincare) waves as well as for electromagnetic wave modes in waveguides and plasma waves was initiated in the author’s book [10].

There is a rather long story of the method development for acoustics and its effects in the papers of Perelomova [11–24]. The main achievements are in the explicit construction of the projection operators in exponentially stratified and homogeneous fluids in $1+1$, $1+2$, and $1+3$ dimensions with introduction of entropy, acoustic and vortical modes [11, 13, 14, 25]. The investigations also account for a variety of thermodynamic properties such as equations of states. The investigation of their interaction made it possible to describe such important phenomena as heating and streaming for arbitrary form of acoustic fields [16, 19–22]. The study also gives an answer to the important question about the role of the joint action of thermoconductivity, viscosity and nonlinearity. Let us stress also the very interesting development of the theory with a nonlinearity account inside the projection technique, when the fields (modes) are correspondingly

redefined [16]. Let us also mention the famous integrable KdV equation derivation and its solitons in a bubbly liquid [12].

As initiation of a parallel investigation, almost identical to ours, let us mention also the Moscow Nonlinear Acoustics conference paper [26] where an introduction of the vortical and acoustic mode was considered as a starting point to the nonlinear acoustics problem formulation. No development of this start-up has been found till today.

An alternative and promising approach is developed in the group of [27], combined with an averaging procedure [28] and with interesting application to the generalized Foldy-Wouthuysen transformation [29]. It includes a realm of problems over non-homogeneous background using pseudodifferential operators as a tool.

An important step of the theory development in the direction of problems with inhomogeneous background has been made in [30–33] where the projection technique has been built for equations with variable coefficients. In such case it is only the exponential dependence that allows a problem to be reformulated in terms of equations with constant coefficients [11, 13, 14], hence – to apply the method directly. Generally speaking, we should either rewrite the evolution operator in discrete form, expanding the problem variables in some functional basis, or fix the subspace in some physically reasonable differential form. A good example of such action is known in classical hydrodynamics as one that fixes a flow with the velocity field \vec{v} either as solenoidal, by $\text{div}\vec{v}=0$, or as non-vortical, or potential, by $\text{rot}\vec{v}=0$. The problem within the projection approach in this case is in finding additional projection conditions, see, *e.g.* [15].

Some important details of the projection operators technique for the planetary Poincare and Rossby waves in the atmosphere have been presented recently in [7, 34].

In conditions of electrodynamics, as mentioned, the approach has been presented in the book [10]. Its important component related to the direct wave separation is investigated in abundant works on unidirectional propagation [26, 35–38], where hybrid electric-magnetic amplitudes have been introduced. Its dynamic projection operator versions are created in [39, 40] that unify the idea of the “hybridization” field with a combination of equations that split the evolution description into modes in the linear version or introduce interaction of the same fields in the nonlinear case. The main feature of the electromagnetic wave propagation in matter is the presence of dispersion of time or space [41]. Its account is most complete and demonstrates many interesting features in the case of metamaterials [42, 43]. The 1+3 electrodynamics implies a combination of some basic expansion to account for a symmetry of a problem and, having already a one-dimensional subproblem, to apply the projectors that fix as a polarization mode as well as the direction of propagation [44, 45].

The wave identification problem is important in physics of atmosphere, or, more generally, in geophysical hydrodynamics where superposition of acoustic,

gravity and planetary waves takes place over some slow varying dynamics [46]. In the planetary range the scales of Rossby and Poincare waves are of the same order, which also complicates their distinguishing. Similar problems are met in electrodynamics and plasma physics that are even more rich in dynamics properties. Generally, in geophysics, the wave field diagnostics needs many observations that would cover a space sufficient for wave length estimation. It is rather expensive and sometimes not very feasible.

The projection method allows identifying waves in quite a different way. We could measure not the space or time characteristics, but relations between components of the state vector (hydrodynamic field variables in hydrodynamics or electric and magnetic field components in electrodynamics). Such relation is unique for each wave type (mode). In the electromagnetic case such relations are conventionally named polarization relations, but their significance is wider and includes, for example, the direction of propagation. Hence, the diagnostics may be delivered at a space point vicinity. We spread the term “polarization” to be used for all dynamic descriptions of interest as hydrodynamics or plasma.

There are some specific examples that are rigorously studied by mathematicians for the tsunami problem [47]. It is proved that the initial state is uniquely determined by the mode of vibration of one of the points on the surface of the ocean. We suggest to use measurements in a vicinity of a point but with many-component observations with the aid of the projection operator technique, such that fits the subspaces of a specific wave [48].

The main idea of wave diagnostics may be understood from the simplest example of a $1+1$ wave equation (widely known as the string equation) [10, 47]. In this $1+1$ case the wave type (say – “polarization”) is linked to the direction of propagation that allows the whole algorithm of a diagnostic problem solution to be formulated in the following form [48]:

- (1) Projection to subspaces and their weight evaluation in an appropriate physical norm.
- (2) Time arrival and form recognition for a given number of measurements.
- (3) Estimation of the distance to the area of initialization.
- (4) Investigation of the dataset stability in terms of explicit solution form, reconstructed by the finite point data.

It should be mentioned that the last point relates to the analytical continuation problem. Such formulation is very close to the so-called inverse problem of wave initialization [49].

To fit the ordinary article volume we restrict ourselves only by $1+1$ dimension problems that, however, contain all the important components of the approach.

2. General proper space definition – eigenvector problem for perturbations over a homogeneous ground state

2.1. Statement of the Cauchy problem

We start from the mathematical statement of a linear evolution problem in one-dimensional position coordinate space.

Let the independent variables ranges $t > 0, x \in (-\infty, \infty)$ define a half-space.

Consider the evolution equation

$$\psi_t = L(\partial)\psi \tag{1}$$

for the columnne $\psi(x, t) = (\psi_1, \dots, \psi_n)^T \in A^n$, that is the n -dimensional vector field on the 1+1 half space, while

$$\partial = \partial/\partial x \tag{2}$$

The initial condition for the vector $\psi(x, t)$

$$\psi(x, 0) = \phi(x) \tag{3}$$

defines the Cauchy problem to be solved.

2.2. Solution in the k -domain

2.2.1. Transition to the k -domain

Let the Fourier transformation operator be denoted as F

$$F\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[-ikx] \psi(x) dx = \tilde{\psi}(k, t) \tag{4}$$

that acts on each component and its inverse is marked as F^{-1} ,

$$F^{-1}\tilde{\psi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp[ikx] \tilde{\psi}(k) dk \tag{5}$$

Note, that for physical variables the components are real, hence

$$\tilde{\psi}^*(-k) = \tilde{\psi}(k) \tag{6}$$

Applying transformation F to both sides of (1), inserting the identity operator yields the equation

$$F\psi_t = FL(\partial)F^{-1}F\psi$$

that leads to a system of ordinary differential equations

$$L(\tilde{k})\tilde{\psi} = \tilde{\psi}_t \tag{7}$$

where $L(\tilde{k}) = FL(\partial)F^{-1}$, $F\psi = \tilde{\psi}$.

2.2.2. Matrix eigenspaces and projectors

Consider an $n \times n$ matrix eigenvalue problem

$$\tilde{L}\phi = \lambda\phi \quad (8)$$

that introduces n subspaces, which we represent by the matrix of solutions Ψ

$$\tilde{L}\Psi = \Psi\Lambda \quad (9)$$

with the diagonal matrix $\Lambda = \text{diag}\{\lambda_1, \dots, \lambda_n\}$. We would choose the normalization of the eigenvectors such that the first component of each column is unit

$$\Psi_{1j} = 1 \quad (10)$$

By the choice of linear independent eigenvectors, the inverse matrix Ψ^{-1} exists and

$$\Psi^{-1}\tilde{L} = \Lambda\Psi^{-1} \quad (11)$$

Multiplying (9) from the left side by Ψ^{-1} gives

$$\tilde{L} = \Psi\Lambda\Psi^{-1} \quad (12)$$

or, in components, it gives spectral decomposition of matrix \tilde{L}

$$\tilde{L}_{ij} = \Psi_{ik}\Lambda_{kl}\Psi_{lj}^{-1} = \Psi_{ik}\lambda_k\Psi_{kj}^{-1} = \sum_k \lambda_k \Psi_{ik}\Psi_{kj}^{-1} = \sum_s \lambda_s (\tilde{P}^s)_{ij} \quad (13)$$

Where

$$(\tilde{P}^s)_{ij} = \Psi_{is}\Psi_{sj}^{-1} \quad (14)$$

are the projection operators in k -representation.

As a corollary we get basic properties of projectors

$$\sum_{s=1}^n (\tilde{P}^s)_{ij} = \sum_{s=1}^n \Psi_{is}\Psi_{sj}^{-1} = \delta_{ij} \quad \Psi_{is}\Psi_{sj}^{-1}\Psi_{jt}\Psi_{tk}^{-1} = \Psi_{jt}\Psi_{tk}^{-1} \quad (15)$$

that reads as

$$\sum_{s=1}^n \tilde{P}^s = I \quad \tilde{P}^s \tilde{P}^s = \tilde{P}^s \quad (16)$$

Next, we formulate the evolution operator diagonalization theorem for x -independent coefficients.

For the matrix $\tilde{L} = FLF^{-1}$, parameterized by the wavenumber k it is necessary and sufficient that \tilde{L} is equivalent to

$$\tilde{L} = \sum_s \lambda_s(k) \tilde{P}^s(k) \quad (17)$$

where λ_i is given by a root of the algebraic (dispersion) equation

$$\det(\tilde{L} - \lambda I) = 0 \quad (18)$$

and $P^i P^k = \delta_{ik} \tilde{P}^k$.

The system (7) splits as

$$\tilde{L}(\tilde{P}^i \tilde{\psi}) = (\tilde{P}^i \tilde{\psi})_t \tag{19}$$

For a proof it is enough to commute

$$\tilde{P}^i \tilde{L} = \tilde{L} \tilde{P}^i \tag{20}$$

that follows from the spectral decomposition (8):

$$\tilde{P}^i \tilde{L} = \sum_s \lambda_s \tilde{P}^i \tilde{P}^s = \lambda_i \tilde{P}^i = \sum_s \lambda_s \tilde{P}^s \tilde{P}^i \tag{21}$$

2.2.3. Subspaces evolution in the k -domain

Then, the Equation (19) reads as

$$\tilde{L}(\tilde{P}^i \tilde{\psi}) = (\tilde{P}^i \tilde{\psi})_t = \lambda_i \tilde{P}^i \tilde{\psi} \tag{22}$$

with the solution

$$\tilde{P}^i(k) \tilde{\psi}(k, t) = C_i(k) \exp[\lambda_i(k)t] \tag{23}$$

where $C_i(k)$ is defined by the initial condition

$$\tilde{P}^i(k) \tilde{\psi}(k, 0) = C_i(k) \tag{24}$$

The general solution of (7), due to the evolution operator linearity, is the sum of the particular solutions (23)

$$\sum_{i=1}^n \tilde{P}^i(k) \tilde{\psi}(k, t) = \tilde{\psi}(k, t) = \sum_{i=1}^n C_i(k) \exp[\lambda_i(k)t] \tag{25}$$

if $\lambda_i \neq \lambda_k$ due to the completeness property of projectors (16). Having in mind the condition (6), we write

$$\tilde{\psi}^*(-k, t) = \sum_{i=1}^n C_i^*(-k) \exp[\lambda_i^*(-k)t] = \tilde{\psi}(k, t) \tag{26}$$

so the reality of the solution implies conditions on the particular solutions and evolution operator spectrum (dispersion relation). In the case of the choice

$$\lambda_i^*(-k) = \lambda_i(k) \tag{27}$$

one has

$$C_i^*(-k) = C_i(k), \tag{28}$$

that is necessary to take into account constructing the directed or other fields in x - or t -domains.

Due to the matrix of solutions Ψ normalization choice (10), it is convenient to introduce new variables

$$\tilde{P}_{1j}^s(k)\tilde{\psi}_j(k,t) = \Psi_{1s}\Psi_{sj}^{-1}\psi_j(k,t) = \tilde{\Pi}^s(k,t) \quad (29)$$

The explicit form of the matrix \tilde{P}^i allows us to write

$$\Psi_{2s}\Psi_{sj}^{-1}\tilde{\psi}_j(k,t) = \Psi_{2s}\Psi_{1s}^{-1}\Psi_{1s}\Psi_{sj}^{-1}\tilde{\psi}_j(k,t) = \Psi_{2s}\Psi_{1s}^{-1}\tilde{\Pi}^s(k,t) \quad (30)$$

and, similar for the indexes $3, \dots, n$. Summation of all such equations by s starting from (29) gives a system of n algebraic equations for $r = 1, \dots, n$

$$\sum_{s=1}^n \tilde{P}_{rj}^s(k)\tilde{\psi}_j(k,t) = \delta_{rj}\tilde{\psi}_j(k,t) = \tilde{\psi}_r = \sum_{s=1}^n \Psi_{rs}\Psi_{1s}^{-1}\tilde{\Pi}^s(k,t) \quad (31)$$

as a corollary of the completeness of the set of P^s (15). The result gives the explicit form of the Fourier transforms of all physical variables – components of vector $\tilde{\psi}_j(k,t)$. The simplest expression is obtained for $\tilde{\psi}_1$:

$$\tilde{\psi}_1 = \sum_{s=1}^n \tilde{\Pi}^s(k,t) \quad (32)$$

Solving the system (31) with respect to $\tilde{\Pi}^s(k,t)$, one obtains the inverse transformation, expressing $\tilde{\Pi}^s$ as a linear function of the principle components transform $\tilde{\psi}_j(k,t)$.

3. Transition to x -representation

3.1. Homogeneous equation

After the inverse Fourier transform, the original system (1) splits as

$$(P^i\psi)_t = LP^i\psi \quad (33)$$

where the formula $P^i = F^{-1}\tilde{P}^iF$ represents the matrix integral operator of the convolution type, so that

$$(F^{-1}\tilde{P}_{jk}^iF)\phi(x) = P_{jk}^i\phi(x) = \int_{-\infty}^{\infty} K_{jk}^i(x-y)\phi(y)dy \quad (34)$$

with the kernel

$$K_{jk}^i(x-y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{P}_{jk}^i(k) \exp[ik(x-y)] dk \quad (35)$$

Let us check the statement. The transformations sequence

$$(\tilde{P}^i\tilde{\psi})_t = (\tilde{P}^iF\psi)_t = FLF^{-1}\tilde{P}^i\tilde{\psi} = FLF^{-1}\tilde{P}^iF\psi \quad (36)$$

proves the resulting formula (33). The splitting is more obvious, if one performs substitutions based on the identity

$$\sum_{i=1}^n P^i\psi = \psi \quad (37)$$

that is the transform of (15), it reads as a system of equations for the components transition to new variables

$$\sum_{j=1}^n P_{kj}^i \psi_j = \Pi_k^i \quad k = 1, \dots, n \tag{38}$$

Namely, we can fix the choice of the new variables as $\Pi^i = P_{1j}^i \psi_j$ from the relation (38), or

$$\Pi^i(x, t) = P_{1k}^i \psi_k(x, t) = \int_{-\infty}^{\infty} K_{1k}^i(x-y) \psi_k(y, t) dy \tag{39}$$

that is a one-to-one map.

So, we check the statement by means of the definition of projectors in k -space (14)

$$\sum_{s=1}^n \tilde{\Pi}^s = \sum_{s=1}^n \Psi_{1s} \Psi_{sj}^{-1} \phi_j = \delta_{1j} \phi_j = \phi_1 \tag{40}$$

other relations are transforms of (31) with $k = 2, 3, \dots, n$, that gives an equivalent description in the x -representation.

$$\sum_{s=1}^n F^{-1} \tilde{P}_{rj}^s(k) F F^{-1} \tilde{\psi}_j(k, t) = F^{-1} \tilde{\psi}_r = \sum_{s=1}^n F^{-1} \Psi_{rs} \Psi_{1s}^{-1} F F^{-1} \tilde{\Pi}^s(k, t) \tag{41}$$

In the integral form one has the inverse transform

$$\psi_r(x, t) = \sum_{s=1}^n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_{rs}(k, t) \Psi_{1s}^{-1}(k, t) \exp[ik(x-y)] dk \Pi^s(y, t) dy \tag{42}$$

that completes the proof of the statement.

The initial conditions for the mode variables $\Pi^s(y, 0)$ are extracted from the identity $\sum_1^n P^i \phi = \phi$, for a given vector ϕ from (3) or, taking (42) at $t = 0$

$$\psi_r(x, 0) = \phi_r(x) = \sum_{s=1}^n \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_{rs}(k, t) \Psi_{1s}^{-1}(k, t) \exp[ik(x-y)] dk \Pi^s(y, 0) dy \tag{43}$$

that is solved with respect to $\Pi^s(y, 0)$ by (39).

Finally we arrive at

$$\Pi^i(x, 0) = P_{1k}^i \psi_k(x, 0) = \int_{-\infty}^{\infty} K_{1k}^i(x-y) \phi_k(y) dy \tag{44}$$

that accomplishes the problem formulation for the mode variables Π^s together with the x -representation (45) for the equation (33):

$$\lambda_s(\partial) \Pi^s = \Pi_t^s \tag{45}$$

The problem is, in fact, solved by the sequence of formulas (23) (24) after the inverse Fourier transform, but its nonlinear version (see the Section 5) is a nontrivial generalization that usually, cannot be solved by the Fourier transformation method.

3.2. Inhomogeneous equation

Consider a problem with an r.h.s of the evolution equation as a vector determined as a source. Many modeling problems of interest describe the action of the source by means of the vector $f(x,t) = (f_1, \dots, f_n)^T$

$$\psi_t - L(\partial)\psi = f(x,t) \quad (46)$$

each component of which represents the action of the corresponding (e.g. heat, momentum, mass) source. The natural application of the dynamic projection operators is the action as in (33)

$$(P^i \psi)_t - L(\partial)P^i \psi = P^i f(x,t) \quad (47)$$

(45) is modified as

$$\Pi_t^i - \lambda^i(\partial)\Pi^i = P_{1j}^i f_j(x,t) = \int_{-\infty}^{\infty} K_{1j}^i(x-y) f_j(y,t) dy \quad (48)$$

because of the definition (39).

4. Boundary regime propagation

4.1. Problem reformulation

The geometry of an experiment often needs a transition from the Cauchy problem formulation to a boundary regime one by specific functions of time for each variable at some fixed point usually chosen at $x = 0$. Such situation is realized, if we have a plane wave falling at a plane interface between two media. Similarly a waveguide mode is excited by some perturbation at an end of the waveguide. Such a 1+1 problem may be written in the form, for an equation quite similar to the basic system (1), but after rearranging as

$$\partial_x \psi = L(\partial_t) \psi \quad (49)$$

for a vector $\psi(x,t)$ generally of the same or higher number of components with the boundary condition

$$\psi(0,t) = \phi(t) \quad (50)$$

A reformulation of Equation (1) in the case when we have only x -derivatives of the first order in $L(\partial_x)$ is simple – we shift the terms with time derivative to the r.h.s. and vice versa for x -derivatives. The result has the r.h.s. in the form of $L\psi$ automatically. If the operator L contains second or higher derivatives, sometimes one can manipulate with the differentiation and arrangement of the terms till the form of (49) is achieved. Generally the known trick is applied: having the second derivative, say a_{xx} , we denote $a_x = b$, having

$$\partial_x b = \dots \quad (51)$$

and, consider the notations as new equations

$$\partial_x a = b \quad (52)$$

The order of the system grows from n to $n+1$ at each such step.

For example, if we start from a diffusion equation for a concentration $u(x, t)$ and the diffusion coefficient D ,

$$u_t = Du_{xx} \tag{53}$$

with the given boundary values $u(0, t) = \mu(t)$ and the mass flow proportional to $u_x(0, t) = \nu(t)$. Considering it as the system of the order 1, we apply the mentioned trick arriving at

$$\begin{aligned} v_x &= D^{-1}u_t \\ u_x &= v \end{aligned} \tag{54}$$

or, in the form of Equation (49)

$$\partial_x \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ D^{-1}\partial_t & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{55}$$

with the boundary conditions (regime)

$$\begin{pmatrix} u(0, t) \\ v(0, t) \end{pmatrix} = \begin{pmatrix} \mu(t) \\ \nu(t) \end{pmatrix} \tag{56}$$

4.2. The problem solution

Now we should deliver a Fourier transformation in t -variable. The natural obstacle is the $t, x \leq 0$ domain under consideration. The continuation of the function u, v to the negative time values may be realized taking the boundary conditions (56) structure into account. For physics reasons the switching type conditions may be chosen so that u and v have zero values for $t=0$, that means the choice of odd functions with respect to the reflection $t \rightarrow -t$. Hence, the Fourier transformation reads as substitution

$$u(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}(x, \omega) \exp[i\omega t] d\omega = F^{-1}\tilde{u}(x, \omega) \tag{57}$$

the reality of function u implies that

$$\begin{aligned} u(x, t)^* &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}^*(x, \omega) \exp[-i\omega t] d\omega = \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}^*(x, -\omega') \exp[i\omega' t] d\omega' \end{aligned} \tag{58}$$

for $\omega = -\omega'$. Then

$$\tilde{u}^*(x, -\omega) = \tilde{u}(x, \omega) \tag{59}$$

The same as with t dependence problem appears for the transforms dependent on ω . The function $\tilde{u}(x, \omega)$ symmetry follows from

$$\begin{aligned} u(x, -t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}(x, \omega) \exp[-i\omega t] d\omega = \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}(x, -\omega') \exp[i\omega' t] d\omega' = -u(x, t) \end{aligned} \tag{60}$$

after the substitution $\omega = -\omega'$. It yields

$$\tilde{u}(x, -\omega) = -\tilde{u}(x, \omega) \tag{61}$$

So we should account for both (59) and (61). Similarly we transform $v \rightarrow \tilde{v}$, arriving at the ODE system

$$\partial_x \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ i\omega D^{-1} & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \quad (62)$$

The boundary values at each ω point are

$$\begin{pmatrix} \tilde{u}(0, \omega) \\ \tilde{v}(0, \omega) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(0, t) \exp[-i\omega t] dt \\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(0, t) \exp[-i\omega t] dt \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu(t) \exp[-i\omega t] dt \\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \nu(t) \exp[-i\omega t] dt \end{pmatrix} \quad (63)$$

Now we can apply the projection technique in the spirit of this article. Consider the eigenproblem

$$\begin{pmatrix} 0 & 1 \\ i\omega D^{-1} & 0 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \lambda \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \quad (64)$$

that gives eigenvalues

$$\lambda_{\pm} = \pm \sqrt{i\omega D^{-1}} \quad (65)$$

and eigenvectors

$$\phi_+ = \begin{pmatrix} 1 \\ \lambda_+ \end{pmatrix} \quad \phi_- = \begin{pmatrix} 1 \\ \lambda_- \end{pmatrix} \quad (66)$$

Then

$$\Psi = \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix} \quad (67)$$

and

$$\Psi^{-1} = \frac{1}{\lambda_- - \lambda_+} \begin{pmatrix} \lambda_- & -1 \\ -\lambda_+ & 1 \end{pmatrix} \quad (68)$$

that yields the projection matrices in ω -representation applying Equation (14).

In our case

$$\lambda_- = -\lambda_+$$

therefore

$$\Psi^{-1} = \frac{1}{2} \begin{pmatrix} 1 & \lambda_+^{-1} \\ 1 & -\lambda_+^{-1} \end{pmatrix} \quad (69)$$

The projectors

$$P^{\pm} = \frac{1}{2} \begin{pmatrix} 1 & \pm \lambda_+^{-1} \\ \pm \lambda_+ & 1 \end{pmatrix} \quad (70)$$

acting on Equation (62)

$$\partial_x \begin{pmatrix} 1 & \pm \lambda_+^{-1} \\ \pm \lambda_+ & 1 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} 1 & \pm \lambda_+^{-1} \\ \pm \lambda_+ & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ i\omega D^{-1} & 0 \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} \quad (71)$$

reads along the first line

$$\partial_x (\tilde{u} \pm \lambda_+^{-1} \tilde{v}) = \pm \lambda_+ \tilde{u} + \tilde{v} \quad (72)$$

The second line gives as usually the same (proportionality with some operator coefficient) result. Multiplying (72) by λ_+ one has

$$\partial_x (\lambda_+ \tilde{u} \pm \tilde{v}) = \pm \lambda_+ (\lambda_+ \tilde{u} \pm \tilde{v}) \quad (73)$$

Denoting

$$\tilde{\Pi}_{\pm} = \lambda_+ \tilde{u} \pm \tilde{v} \tag{74}$$

we arrive at the x -evolution equations in ω domain

$$\partial_x \tilde{\Pi}_{\pm} = \pm \lambda_+ \tilde{\Pi}_{\pm} \tag{75}$$

and the boundary conditions

$$\tilde{\Pi}_{\pm}(0, \omega) = \lambda_+ \tilde{u}(0, \omega) \pm \tilde{v}(0, \omega), \tag{76}$$

or

$$\tilde{\Pi}_{\pm}(0, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [\lambda_+ \mu(t) \pm \nu(t)] \exp[-i\omega t] dt \tag{77}$$

with the solution

$$\tilde{\Pi}_{\pm}(x, \omega) = \exp[\pm \lambda_+(\omega)x] \tilde{\Pi}_{\pm}(0, \omega) \tag{78}$$

Now we should return in (75) to the Fourier original via

$$\tilde{u}(x, \omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x, t) \exp[-i\omega t] d\omega = Fu \tag{79}$$

Acting on (75) by F^{-1} from the left side we write

$$\partial_x F^{-1} \tilde{\Pi}_{\pm} = \pm F^{-1} \lambda_+ F F^{-1} \tilde{\Pi}_{\pm} \tag{80}$$

or, denoting $\Pi_{\pm} = F^{-1} \tilde{\Pi}_{\pm}$,

$$\partial_x \Pi_{\pm} = \pm F^{-1} \lambda_+(\omega) F \Pi_{\pm} \tag{81}$$

or, in terms of the time derivative operator ∂_t , we have

$$\partial_x \Pi_{\pm} = \pm \lambda_+(\partial_t) \Pi_{\pm} \tag{82}$$

The solution may be presented either in pseudodifferential form

$$\Pi_{\pm} = \exp[\pm \lambda_+(\partial_t)x] \Pi_{\pm}(0, t) \tag{83}$$

or in the integral one via (78)

$$\Pi_{\pm}(x, t) = F \tilde{\Pi}_{\pm}(x, \omega) = F \exp[\pm \lambda_+(\omega)x] \tilde{\Pi}_{\pm}(0, \omega) \tag{84}$$

Finally, by means of (77) we write

$$\Pi_{\pm}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[\pm \lambda_+(\omega)x] \exp[i\omega(t-\tau)] [\mu(\tau) \lambda_+(\omega) \pm \nu(\tau)] d\tau d\omega \tag{85}$$

or, plugging (65), yields

$$\Pi_{\pm}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[\pm \sqrt{i\omega D^{-1}}x] \exp[i\omega(t-\tau)] [\mu(\tau) \sqrt{i\omega D^{-1}} \pm \nu(\tau)] d\tau d\omega \tag{86}$$

It solves the boundary problem (55) (56) by means of the inverse of the t -version of (74):

$$\Pi_{\pm} = \lambda_+(\partial_t) u \pm v \tag{87}$$

The inverse transform gives the original variable (concentration)

$$u = \frac{1}{2} \lambda_+ (\partial_t)^{-1} (\Pi_+ + \Pi_-) \quad (88)$$

The variable v is expressed as

$$v = \frac{1}{2} (\Pi_+ - \Pi_-) \quad (89)$$

We would recall that the variable v originated from $v = u_x$, so there is a link between (88) and (89). A heat transfer problem, including the celebrated heat waves, is solved almost identically, plugging the heat conductivity coefficient instead of the diffusion coefficient D .

5. On weak nonlinearity account problems

Let again the independent variables ranges such as $t > 0$, $x \in (-\infty, \infty)$ define a half-space and the evolution equation

$$\psi_t - L(\partial)\psi = \epsilon N(\psi) \quad (90)$$

for $\psi(x, t) \in A^n$, contains now the so-called weak nonlinear terms $N(\psi)$, the small parameter ϵ appears after rescaling of the wave vector $\psi \rightarrow \epsilon\psi$.

The initial condition for the vector

$$\psi(x, 0) = \phi(x) \quad (91)$$

defines the Cauchy problem to be solved.

The formal idea to make a first step to reformulate the problem in terms of the linear modes introduced in the previous sections is based on the substitution (43). In such terms the l.h.s. splits and allows us to apply the procedure of perturbation expansion of the equation operator in the power series with respect to the parameter ϵ [10]. The procedure is performed by the action of the projection operators on the equation (90):

$$P^s(\psi_t - L(\partial)\psi) = \epsilon P^s N \left(\sum_{j=1}^n P^j \psi \right) \quad (92)$$

the unit operator is inserted in the nonlinear part of the equation to express the r.h.s. in terms of transformed field variables. It has been proved that the projection operators commute with the linear evolution operator L by construction (see the Section 3), therefore

$$(P^s \psi)_t - L(\partial)P^s \psi = \epsilon P^s N \left(\sum_{j=1}^n P^j \psi \right) \quad (93)$$

The result of the operator P^s application to ψ gives (39), so the first line of Equation (93) reads as

$$\Pi_t^s - \lambda_s(\partial)\Pi^s = \epsilon P_{1j}^s N_j \left(\sum_{j=1}^n P^j \psi \right) \quad (94)$$

The components of the nonlinear r.h.s of the system may be understood as

$$N_j \left(\sum_{j=1}^n P^j \psi \right) = N_j(\psi_1, \dots, \psi_n) \tag{95}$$

where the components of the basic vector ψ_i are expressed in terms of the mode amplitudes Π^s via Equation (43). Plugging Equation (95) into (94) we obtain a system of equations that describes interactions of the principal modes. The l.h.s. of the system may be considered as a result of the so-called diagonalization of a basic system like general hydrodynamic or electrodynamics.

The initial conditions for $\Pi^s(x, t)$ are determined by the projection procedure (44), or

$$\Pi^i(x, 0) = \int_{-\infty}^{\infty} K_{1k}^i(x-y)\phi_k(y)dy = \chi^i(x) \tag{96}$$

If a set of initial conditions (91) is chosen as

$$\chi^i(x) = \delta_{i1}u(x) \tag{97}$$

only one mode is excited, we can neglect the other modes generation till time $T \propto \epsilon^{-1}$ and operate by the equation for $U(x, t) = \Pi^1(x, t)$:

$$U_t - \lambda_1(\partial)U = \epsilon P_{1j}^1 N_j(\psi_1, \dots, \psi_n) \tag{98}$$

The expression for ψ_i as the function of U is given by (43)

$$\psi_r(x, t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi_{r1}(k, t)\Psi_{11}^{-1}(k, t)\exp[ik(x-y)]dkU(y, t)dy \tag{99}$$

Such equation describes the self-action of the first mode. Similar equations may be written for each mode function Π^2, \dots, Π^n .

The integral operators that appear in the theory may be considered as pseudodifferential if we expand the kernels in the Taylor series with respect to k . Such expansion is used to build an approximation, if we restrict ourselves by a wave process the spectrum of which(wavelength values) is located by some range. Such location is controlled by class C of initial conditions $\phi(x) \in C$ and a correspondent small parameter that appears after rescaling of basic variables. The scales for the Cauchy problem may be introduced as follows. Suppose we have the only variable $u(x, t)$, then, if

$$\max|u(x, 0)| = \max_C|\phi(x)| = u_0 \tag{100}$$

and

$$\max_C|u_x(x, 0)| = \max|\phi_x(x)| = k_0u_0 \tag{101}$$

The space scale may be chosen as $\lambda = 2\pi k_0^{-1}$. The time scale choice is defined by the evolution itself as $\tau = 2\pi\omega^{-1}(k_0)$.

In the case of a multicomponent state description, the amplitude parameter is determined via the appropriate vector norm

$$\max_C \|\phi(x, 0)\| = u_0 \quad (102)$$

and the derivatives are estimated as

$$\max_C \|\phi_x(x)\| = k_0 u_0 \quad (103)$$

the evolution of mode s is described by (45), that comes from (8). Therefore, in a conventional relation $\lambda_s = i\omega_s$ expansions of the kernels may be cut at different powers dependent on the time evolution scale

$$\tau_s = 2\pi\omega_s^{-1}(k_0) \quad (104)$$

for a given mode Π^s , because of a different small parameter definition.

The formal solution of Equation (45)

$$\Pi^s(x, t) = \exp[\lambda_s(\partial)t]\Pi(x, 0) \quad (105)$$

yields an estimation of the l.h.s.

$$|\Pi^s(x, t)| \leq \|\exp[\lambda_s(\partial)]\| |\Pi(x, 0)| \quad (106)$$

where $\|A(\partial)\|$ is the pseudodifferential operator $A(\partial)$ norm [28].

In such case we approximate the expansion by a finite number of terms – a polynomial in k , that allows us to change the integral operators by differential ones.

In such a way such famous equations as KdV and Burgers are derived, if only quadratic terms are left in the nonlinear part of the basic system.

6. Conclusion

The universality of the method is explained by its direct application to any system of differential equations, also with sources, in $1+1$ with constant coefficients. There is a possibility to widen its application for systems with variable coefficients in conditions of the presence of a small parameter [48, 27]. The parameter admits expansion therein and the expansion is terminated at a term of some parameter power, considering long or short waves systematically.

Acknowledgements

This work was supported by the Ministry of Education and Science of the Russian Federation (Contract No. 3.1127.2014/K)

References

- [1] Fock V A 1936 *UFN* **16** (8) 1070
- [2] Kovasznay L S G 1953 *J. Aero. Sci.* **20** 657; Chu B-T, Kovasznay L S G 1958 *J. Fluid Mech.* **3** 494
- [3] Novikov A A 1976 *Radiophysics and Quantum Electronics* **19** 225
- [4] Novikov A A 1981 *Waves and diffraction* **2** 66
- [5] Fleck J A Jr. 1970 *Phys. Rev. B* **1** 84

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- [6] Vereshchagina I 1980 *Interaction of modes in multidimensional medium with dispersion*, M. Sc. Thesis, Kaliningrad State University (Im. Kant Federal University from 2010)
- [7] Bessarab F, Karpov I and Leble S 2008 *Projection operators for the planetary Poincare and Rossby waves in the atmosphere*, Vestnik BFU im. Kanta
- [8] Bessarab F, Kshevetskij S and Leble S 1987 *On the Acoustic-Gravity waves Interaction in Atmosphere. Problems of Nonlinear Acoustics*, in: Proceedings of IUPAP, IUTAM symposium on Nonlinear Acoustics Ed. Kedrinskii V K, AN USSR, Siberian Division
- [9] Leble S B and Zaitsev A A 1987 *Novye Methody v Teorii Nelinejnykh Voln*, Kaliningradskij Universitet [Kaliningrad University Press] [New Methods in Nonlinear Waves Theory] (in Russian)
- [10] Leble S B 1988 *Waveguide Propagation of Nonlinear Waves in Stratified Media*, Leningrad University Press (in Russian); 1990 extended ed. in Springer-Verlag
- [11] Perelomova A 1993 *Izv. RAN, Physics of Atm. Ocean* **29** (1) 47
- [12] Perelomova A A 2000 *Applied Mathematics Letters* **13** 93
- [13] Perelomova A A 2010 *Canadian Journal of Physics* **88** (4) 293
- [14] Perelomova A A 2000 *Archives of Acoustics* **25** 451
- [15] Brezhnev Yu V, Leble S B and Perelomova A A 1993 *Physical Express* **41** 29
- [16] Perelomova A 2006 *Physics Letters A* **357** 42
- [17] Perelomova A 2009 *Archives of Acoustics* **34** (2) 127
- [18] Perelomova A 2009 *Ultrasonics* **49** 583
- [19] Perelomova A 2010 *Acta Acustica united with Acustica* **96** 43
- [20] Perelomova A and Pelc-Garska W 2010 *Central European Journal of Physics* **8** (6) 855
- [21] Perelomova A and Wojda P 2011 *Central European Journal of Physics* **9** (3) 740
- [22] Perelomova A and Wojda P 2012 *Central European Journal of Physics* **10** (5) 1116
- [23] Perelomova A 2013 *Acta Physica Polonica A* **123** (4) 681
- [24] Perelomova A 2013 *Acta Acustica united with Acustica* **99** 352
- [25] Perelomova A A 1998 *Acta Acustica* **84** (6) 1002
- [26] Pitois S, Millot G and Wabnitz S 2001 *J. Opt. Soc. Am. B* **18** (4)
- [27] Belov V V, Dobrohotov S Yu and Tudorovskiy T Ya 2006 *Journal of Engineering Mathematics* **55** (1–4) 183
- [28] Brüning J, Grushin V V and Dobrohotov S Yu 2012 *Mathematical Notes* **92** (1) 151
- [29] Brüning J, Grushin V V, Dobrohotov S Yu and Tudorovskii T Ya 2011 *Theoretical and Mathematical Physics* **167** 547
- [30] Leble S 1988 *Nonlinear Waves in Boundary Layer and Turbulence*, in: Theses of XX conference Kaliningrad State University 185
- [31] Leble S B, Popov I Yu and Gugel' Yu V 1993 *Gidromekh.* **67** 3 (in russian)
- [32] Perelomova A and Leble S 2002 *Tollmien-Shlichting and Sound Waves Interaction: Nonlinear Resonances*, in *Nonlinear Acoustics at the Beginning of the 21st Century* ed. Rudenko O V *et al.* (MSU) **1** 203
- [33] Perelomova A and Leble S 2005 *TMF* **144** 1030
- [34] Lebedkina N, Karpov I and Leble S 2012 *A method to construct the projection operators for the Poincare and Rossby waves in the atmosphere*, Vestnik BFU im. Kanta, vyp. 4, Ser. Fiz.-mat. nauki 21
- [35] Kinsler P, Radnor S B P and New G H C 2005 *Phys. Rev. A* **72**, 063807
- [36] Kinsler P 2007 *J. Opt. Soc. Am. B* **24** 2363
- [37] Kinsler P 2010 *Phys. Rev. A* **81**, 013819
- [38] Kinsler P 2010 *Phys. Rev. A* **81**, 023808
- [39] Kuszner M and Leble S 2011 *J. Phys. Soc. Jpn.* **80**, 024002
- [40] Kuszner M and Leble S 2014 *J. Phys. Soc. Jpn.* **83**, 034005
- [41] Caloz Ch *Metamaterial-Based Electromagnetic Space, Time and Spacetime Dispersion Engineering*, IEEE, Advanced technology for Humanity

- [42] Ampilogov D and Leble S 2015 *arXiv:1512.01682v1* (see also this paper)
- [43] Ampilogov D and Leble S 2016 *Physics Letters A* **380** 2271
- [44] Kuszner M, Leble S, Kuszner M, Leble S and Reichel B 2011 *Theoretical and Mathematical Physics* **168** (1) 977
- [45] Kuszner M and Leble S 2015 *Waveguide Electromagnetic Pulse Dynamics: Projecting Operators Method*, in: *Odyssey of Light in Nonlinear Optical Fibers: Theory and Applications* ed. Porsezian K., Ganapathy R, CRC Press Reference
- [46] Pedlosky J 1992 *Geophysical Fluid Dynamics*, Springer
- [47] Romanov V G and Moshkalev P 2011 *Sib. Zh. Industrie. Mat.* **14** (3) 87 (in russian)
- [48] Leble S and Vereshchagina I *The method of dynamic projection operators in the theory of hyperbolic systems of partial differential equations with variable coefficients*, *arXiv:1403.7751*
- [49] Lavrent'ev M M, Reznitskiy K G and Yakhno V G 1986 *American Mathematical Society Translations, series 2* **130**