PROJECTING PROCEDURE FOR METAMATERIAL CYLINDRICAL WAVEGUIDE

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Abstract: In this paper a new way of derivation of an evolution equation for short pulses in a dielectric waveguide including one model of a metamaterial waveguide is shown. This derivation model relies upon projecting to an orthogonal basis. In our case such orthogonal basis for cylindrical waveguides is chosen as Bessel functions.

Keywords: Negative index metamaterial, nonlinearity, NLSE

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1. Introduction

Metamaterials have been in the center of interest during the last decade [1–3]. Specific properties of metamaterials such as negative electric permittivity and magnetic permeability lead to very interesting phenomena and make the researcher focus on intensive studies in this field. A promising metamaterial in the fibers technology is the negative refraction index material (NIM) or the left handed material (LHM) [4]. This kind of a metamaterial could be successfully used as a waveguide for light. There are a lot of works focused on the linear properties of this kind of waveguides and their properties like high dispersion, negative refraction, superlensing [5], reverse Goos-Hänchen shift. Nonlinear effects such as: second order harmonic generation (SHG), parametric amplification and propagation of short and ultrashort pulses in metamaterials are observed also in metamaterials with a negative refraction index [6]. These phenomena allow using metamaterials as lenses, optical switches and other optical devices. In this text we would like to focus on the propagation of short and ultrashort pulses in metamaterial fibers, especially in left handed materials. In this field the Nonlinear Schrödinger Equation (NLSE) and its varieties (like a higher order NLSE) describe the propagation of light in fibers. In such investigations [7] of pulse propagation in NIM waveguides most authors take approaches by using a scalar version of the electromagnetic field to solve a problem. This approach could lead to losing significant information about the nonlinear parameters which are very important from the point of view of designing new types of waveguides. In the derivation of equations of light propagation in this kind of waveguides authors could be divided into those who use the slowly varying envelope approximation (SVEA) [8–10] and those focused on the non-SVEA derivation [11]. The latter method allows considering the propagation of ultrashort pulses that have few tens of optical cycles. The most important case for our consideration are NIM fibers including the Kerr type nonlinearities. We establish that these fibers have nonlinearities of both types coming from magnetization and polarization [10].

Our work is based on the projection procedure [12] which allows including a 2D/3D geometry of a waveguide. In this paper standard permittivity and permeability are used but there is a possibility to include other forms e.g. in the form of bi-isotropic constitutive relations [4].

The following subsections of this section describe the notations with which we work and define the basic equations and material parameters. The second section presents a projecting procedure which is used to derive the propagation equation and the third section shows the derivation of a full vector field.

1.1. Linear part of equations

In this paragraph we define the form of the Maxwell equation which is used in our calculation

$$\nabla\cdot\mathbf{B}=0 \qquad \nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \tag{1a}$$

$$\nabla\cdot\mathbf{D}=0 \qquad \nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t} \tag{1b}$$

$$\mathbf{B}=\mu_0\mathbf{H}+\mathbf{M} \qquad \mathbf{M}=\mu_0\chi_m\mathbf{H} \tag{2a}$$

$$\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P} \qquad \mathbf{P}=\varepsilon_0\chi_e\mathbf{E} \tag{2b}$$

where χ_e and χ_m are the electric and magnetic susceptibility, respectively.

We could write

$$\mathbf{B}=\mu_0\mathbf{H}+\mu_0\chi_m\mathbf{H}=\mu_0\mu\mathbf{H} \tag{3a}$$

$$\mathbf{D}=\varepsilon_0\mathbf{E}+\varepsilon_0\chi_e\mathbf{E}=\varepsilon_0\varepsilon\mathbf{E} \tag{3b}$$

where ε is permittivity and μ is permeability.

We also define the operator

$$\square = \triangle - \varepsilon_0 \varepsilon \mu_0 \mu \frac{\partial^2}{\partial t^2} \tag{4}$$

1.2. Nonlinearities coming from metamaterial assumption

We could connect the material parameters χ_e and χ_m with the metamaterial in a standard way [10, 9, 13], this would allow us to write the equations

$$\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E} + \varepsilon_0 \chi_e^{(3)} |\mathbf{E}|^2 \mathbf{E} \tag{5}$$

$$\mathbf{B} = \mu_0 \mu \mathbf{H} + \mu_0 \chi_m^{(3)} |\mathbf{H}|^2 \mathbf{H} \tag{6}$$

and from Maxwell equations:

$$\nabla(\nabla \cdot \mathbf{H}) - \Delta \mathbf{H} = \frac{\partial}{\partial t} \left[\nabla \times \left(\varepsilon_0 \varepsilon \mathbf{E} + \varepsilon_0 \chi_e^{(3)} |\mathbf{E}|^2 \mathbf{E} \right) \right] \tag{7}$$

In this case we establish that $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{H} = 0$ [14]

In the paper [15, 9, 10] the nonlinear part $|\mathbf{E}|^2$ is treated as a constant. In the mentioned papers this approximation is semi-hidden because the authors exchange this in the curl term by ε_{NL} without any comments or with an assumption of weak nonlinearity.

This procedure in the case of the vector field yields

$$\begin{aligned} \square \mathbf{H} = \varepsilon_0 \varepsilon \mu_0 \chi_m^{(3)} \frac{\partial^2}{\partial t^2} |\mathbf{H}|^2 \mathbf{H} + \varepsilon_0 \chi_e^{(3)} \mu_0 \mu \frac{\partial}{\partial t} |\mathbf{E}|^2 \frac{\partial}{\partial t} \mathbf{H} \\ + \varepsilon_0 \chi_e^{(3)} \mu_0 \chi_m^{(3)} \frac{\partial}{\partial t} |\mathbf{E}|^2 \frac{\partial}{\partial t} |\mathbf{H}|^2 \mathbf{H} \end{aligned} \tag{8}$$

Now the projecting procedure could be used to obtain equations for a wave envelope.

2. Projecting procedure

The projecting procedure [12, 16] allows us to derive the equation for an envelope of light pulses propagated in a waveguide. For simplicity, let us focus on cylindrical geometry. Now the ansatz for the z component is defined in the form

$$E_z(r, \varphi, z) = \sum_{l,n} \mathcal{A}_{ln}(z,t) J_l(\alpha_{n,l} r) e^{il\varphi} \tag{9a}$$

$$B_z(r, \varphi, z) = \sum_{l,n} \mathcal{F}_{ln}(z,t) J_l(\alpha_{n,l} r) e^{il\varphi} \tag{9b}$$

$_{ln}$ means that this calculation could be made for a multi-mode fiber, now let us focus on one mode only. The small parameter ϵ is proportional to $\Delta k / 2\alpha \ll 1$ (connected with the wave packet width), σ is a nonlinear parameter and $\sigma \sim \epsilon^2$ [17]. Some authors make the assumption that $\sigma = \epsilon^2$. These parameters are useful during a comparison of the results with the experiment.

In this case the slowly varying envelope approximation (SVEA) [13, 18] is introduced

$$\xi = \sigma z \tag{10a}$$

$$\tau = (t - k'z)\epsilon \tag{10b}$$

$$\mathcal{A}_{ln}(t,z) = \sigma X_{ln}(\tau,\xi) e^{i(\omega t - kz)} \tag{10c}$$

$$\mathcal{F}_{ln}(t,z) = \sigma Y_{ln}(\tau,\xi) e^{i(\omega t - kz)} \tag{10d}$$

Now let us focus on the nonlinear and linear parts of the equation.

2.1. Linear part

In [12, 16] it is shown that projecting to the orthogonal base (for a cylindrical waveguide) procedure can be used, thus, we can write

$$\left(\Box_z + \alpha^2\right)\mathcal{A} = \left(\frac{\sigma\epsilon^2}{c_0^2} - k^2\epsilon^2\sigma\right)\partial_{\tau\tau}X + \sigma^3\partial_{\xi\xi}X + 2k'\sigma^2\epsilon\partial_{\tau\xi}X + 2ik\sigma^2\partial_\xi X \tag{11}$$

If the terms with σ bigger than 2 in power [16] were removed, the linear part of the Coupled Nonlinear Schrodinger Equations would be received, but in the case considered here all the terms will be taken into account.

2.2. Nonlinear part

Let us plug (10) to the nonlinear part of the RHS of equation (8). We do not focus on the calculation of a coefficient (some of the integrals over the Bessel functions from the orthogonal base are already calculated in [12]). This brings us to

$$\sigma^3 q_1 |X|^2 X + \sigma^3 q_2 |Y|^2 X + \sigma^5 |X|^2 |Y|^2 X \tag{12}$$

2.3. Final equations

Now the linear and nonlinear parts could be written below as equations

$$\begin{aligned}
&\left(\frac{\sigma\epsilon^2}{c_0^2} - k^2\epsilon^2\sigma\right)\partial_{\tau\tau}X + \sigma^3\partial_{\xi\xi}X + 2k'\sigma^2\epsilon\partial_{\tau\xi}X + 2ik\sigma^2\partial_\xi X = \\
&\qquad\qquad \sigma^3 q_1 |X|^2 X + \sigma^3 q_2 |Y|^2 X + \sigma^5 q_3 |X|^2 |Y|^2 X \\
&\left(\frac{\sigma\epsilon^2}{c_0^2} - k^2\epsilon^2\sigma\right)\partial_{\tau\tau}Y + \sigma^3\partial_{\xi\xi}Y + 2k'\sigma^2\epsilon\partial_{\tau\xi}Y + 2ik\sigma^2\partial_\xi Y = \\
&\qquad\qquad \sigma^3 q_2 |Y|^2 Y + \sigma^3 q_1 |X|^2 Y + \sigma^5 q_3 |Y|^2 |X|^2 Y
\end{aligned} \tag{13}$$

These equations look like Coupled Nonlinear Schrödinger equations with additional terms. The last LHS equation has a higher order than the others. But at this, the term σ is in the fifth order, which means that this effect is weak. Coupling between X and Y corresponding to the electric and magnetic field is also obtained, both effects coming from the assumptions used. In the case of scalar equations this procedure could give a correct result but when a full vector field is included, the mentioned assumption for the nonlinear part should not be used because it is impossible to calculate proper nonlinear coefficients in this case.

3. Derivation Without Assumption

Making an assumption for $|\mathbf{E}|^2$ and treating it as a constant like in other papers, but using an orthogonal basis and an ansatz for it in case of a nonlinear effect, the components of vector $\mathbf{E}$ in cylindrical coordinates could be written as

$$E_z(r,\varphi,z) = \sum_{l,n} \mathcal{A}_{ln}(z,t) J_l(\alpha_{n,l} r) e^{il\varphi} \tag{14}$$

$$\begin{aligned} E_r(r,\varphi,z) = &-\sum_{l,n} \frac{i}{\alpha_{n,l}^2} \left[\tilde{\mathcal{B}}_{ln}(z,t) \frac{il\omega}{r} J_l(\alpha_{n,l} r) \right] e^{il\varphi} \\ &-\sum_{l,n} \frac{i}{\alpha_{n,l}^2} \left[\tilde{\mathcal{C}}_{ln}(z,t) k_{ln} J_l'(\alpha_{n,l} r) \right] e^{il\varphi} \end{aligned} \tag{15}$$

the remaining components $E_\varphi(r,\varphi,z)$, $B_z(r,\varphi,z)$, $B_r(r,\varphi,z)$, $B_\varphi(r,\varphi,z)$ could be found in [16].

Therefore, the curl could be calculated as

$$\nabla \times \mathbf{E} = \frac{1}{r}\left(\frac{\partial E_z}{\partial \varphi} - r\frac{\partial E_\varphi}{\partial z} \right) \mathbf{e}_r + \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \right) \mathbf{e}_\varphi + \frac{1}{r}\left(\frac{\partial (rE_\varphi)}{\partial r} - \frac{\partial E_r}{\partial \varphi} \right) \mathbf{e}_z \tag{16}$$

and also the term $\nabla \times \left(|\mathbf{E}|^2 \mathbf{E} \right)$ could be calculated.

It was only the z component, as in (14), that was needed to be calculated, and thus the following could be written (using a formula for the curl in cylindrical coordinates) [16]:

$$(\nabla \times (|\mathbf{E}|^2 \mathbf{E}))_z = \frac{1}{r}\left(\frac{\partial (rE_\varphi)}{\partial r} - \frac{\partial E_r}{\partial \varphi} \right) \tag{17}$$

For simplicity, one polarization only is taken into account in this calculation. As the first step we calculate:

$$|\mathbf{E}^{ln}|^2 = E_z^{ln} \overline{E}_z^{ln} + E_r^{ln} \overline{E}_r^{ln} + E_\varphi^{ln} \overline{E}_\varphi^{ln} \tag{18}$$

here, the electric field is written with the number of modes, because generally, the coefficient ("nonlinear coefficient") will depend on the number of modes (which means that it will depend on φ).

For example, for the z component, we have:

$$|\mathbf{E}_z|^2 = |\mathcal{A}|^2 J_l(\alpha r) \tag{19}$$

where J is a Bessel function.

For other components and other modes we have different orders of Bessel functions and on $\exp(il\varphi)$, generally, on the waveguide cross section dimension. In the case of (10) the following equation is obtained

$$\left(\frac{\sigma\epsilon^2}{c_0^2}-k^2\epsilon^2\sigma\right)\partial_{\tau\tau}X+\sigma^3\partial_{\xi\xi}X+2k'\sigma^2\epsilon\partial_{\tau\xi}X+2ik\sigma^2\partial_\xi X=\sigma^3Q_1|X|^2X \tag{20}$$

where Q_1 denotes nonlinear parameters. In this case there is no coupling between the magnetic and electric fields like in the first derivation. Coupling could be obtained only between different modes or different polarizations (sometimes called polarization modes).

If equation (10) is not used, but if the Shafer-Wayne [19] scaling is used for the envelope

$$\xi=\sigma z \tag{21a}$$

$$\tau=(t-k'z)\epsilon \tag{21b}$$

$$\mathcal{A}_{ln}(t,z)=\sigma X_{ln}(t,z) \tag{21c}$$

$$\mathcal{F}_{ln}(t,z)=\sigma Y_{ln}(t,z) \tag{21d}$$

the propagation equation is obtained in the form:

$$\begin{aligned}\left(\frac{\sigma\epsilon^2}{c_0^2}-k^2\epsilon^2\sigma\right)\partial_{\tau\tau}X+\sigma^3\partial_{\xi\xi}X+2k'\sigma^2\epsilon\partial_{\tau\xi}X+2ik\sigma^2\partial_\xi X=\\ \sigma^3Q_1|X|^2X+\sigma^4Q_2|X|^2\partial_\xi X+\sigma^4Q_3|X|^2\partial_\tau X+\\ \sigma^4Q_4|Q_{4a}\partial_\xi X+Q_{4b}\partial_\tau X|^2X+\sigma^4Q_5|Q_{5a}\partial_\xi X+Q_{5b}\partial_\tau X|^2\partial_\xi X+\\ \sigma^4Q_6|Q_{6a}\partial_\xi X+Q_{6b}\partial_\tau X|^2\partial_\tau X\end{aligned} \tag{22}$$

Separated equations for magnetic and electric fields are obtained here and a coupling could be obtained by the Maxwell equations only.

Now let us focus on equations including all the terms

$$\begin{aligned}\nabla\times\nabla\times\mathbf{H}&=\nabla\times\frac{\partial\mathbf{D}}{\partial t}\\ \nabla(\nabla\cdot\mathbf{H})-\Delta\mathbf{H}&=\frac{\partial}{\partial t}\left[\nabla\times\left(\varepsilon_0\varepsilon\mathbf{E}+\varepsilon_0\chi_e^{(3)}|\mathbf{E}|^2\mathbf{E}\right)\right]\end{aligned} \tag{23}$$

The term $\nabla(\nabla\cdot\mathbf{H})$ could be from a Maxwell equation with the divergence of vector $\mathbf{B}$ calculated as shown below

$$\nabla\cdot\mathbf{B}=0 \tag{24}$$

and

$$\mathbf{B}=\mu_0\mu\mathbf{H}+\mu_0\chi_m^{(3)}|\mathbf{H}|^2\mathbf{H} \tag{25}$$

this two equations yield

$$\nabla\cdot\mathbf{B}=\nabla\cdot\mu_0\mu\mathbf{H}+\nabla\cdot\mu_0\chi_m^{(3)}|\mathbf{H}|^2\mathbf{H}=0 \tag{26}$$

which gives

$$\nabla \cdot \mathbf{H} = -\frac{\cdot \chi_m^{(3)}}{\mu} \nabla \cdot |\mathbf{H}|^2 \mathbf{H} \tag{27}$$

The latter term introduces an additional nonlinear magnetic effect. The same procedure could be applied for the term $\nabla(\nabla \cdot \mathbf{E})$. The gradient including equations (14) should be calculated for the full form.

In this case we receive:

$$\begin{aligned}\left[\nabla\left(\nabla \cdot |\mathbf{E}|^2 \mathbf{E}\right)\right]_z &= K_1 \partial_z\left(|\mathcal{A}|^2\right)\partial_z \mathcal{A} + K_1 |\mathcal{A}|^2 \partial_z^2 \mathcal{A} + K_2 \partial_z(|\mathcal{A}|^2)\mathcal{A} + \\ & K_2\left(|\mathcal{A}|^2\right)\partial_z \mathcal{A} + K_3 \partial_z\left(|\partial_z \mathcal{A}|^2\right)\mathcal{A} + K_3\left(|\partial_z \mathcal{A}|^2\right)\partial_z \mathcal{A} + \\ & K_4 \partial_z\left(|\partial_z \mathcal{A}|^2\right)\partial_z \mathcal{A} + K_4\left(|\partial_z \mathcal{A}|^2\right)\partial_z^2 \mathcal{A}\end{aligned} \tag{28}$$

which should be included in the main derivation of the propagation equation.

4. Conclusion

In equation (13) the quintic terms were obtained due to the approximation which was made. It is also possible to cut these terms buy choosing the adequate small nonlinear parameter (σ, ϵ) like in the mentioned papers. However, this method could be insufficient in the case of a full vector field.

New terms in the nonlinear Schrödinger equation (other kind of coupling between equations) are obtained in the presented solution. This case shows only the simplest example (due to the assumption of SVEA) as equation (20). If the slowly varying envelope approximation (SVEA) had not been used (10) but if the Shafer-Wayne [19] approximation had been used, the nonlinear part (22) with the terms with the first derivative along the z direction and the second derivative along the z direction would have been obtained. Unidirectional wave propagation also could be taken into account in this procedure.

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References

[1] Ekici M, Zhou Q, Sonmezoglu A, Moshokoa S P, Ullah M Z, Triki H, Biswas A, Belic M 2017 *Superlattices and Microstructures* (in print, corrected proof)
[2] Ampilogov D, Leble S 2016 *Phys. Lett. A* **38** 2271
[3] Shen L F, Wang Z H 2009 *JOSA A* **24** 754
[4] Zouhdi S, Sihvola A, Vinogradov A P [ed.] 2009 *Handedness in Plasmonics: Electrical Engineer's Perspective in Metamaterials and Plasmonics: Fundamentals, Modelling, Applications*, Springer
[5] Pendry J B 2000 *Phys. Rev. Lett.* **85** 3966
[6] Zharov A A, Shadrivov I V, Kivshar Y S 2003 *Phys. Rev. Lett.* **91** (3) 37401
[7] Maimistov A I, Basharov A M 1999 *Nonlinear optical waves*, Kluwer
[8] Shuangchun W *et al.* 2007 *Phys. Rev. A* **75** 033815-033815-8
[9] Kourakis I, Shukla P K 2005 *Phys. Rev. E* **72** (1) 016626
[10] Tsitsas N L, Rompotis N, Kourakis I, Kevrekidis P G, Frantzeskakis D J 2009 *Phys. Rev. E* **79** (3) 037601

[11] Scalora M, Syrchin M S, Akozbek N, Poliakov E Y, D'Aguanno G, Mattiucci N, Bloemer M J, Zheltikov A M 2005 *Phys. Rev. Lett.* **95** 013902-013902-4
[12] Leble S B, Reichel B 2008 *J. Mod. Opt.* **55** 3651
[13] Syrchin M S, Zheltikov A M, Scalora M 2004 *Phys. Rev. A* **69** (5) 053803
[14] Boyd R W 2003 *Nonlinear optics*, Academic Press
[15] Lazarides N, Tsironis G P 2005 *Phys. Rev. E* **71** (3) 036614
[16] Leble S B, Reichel B 2008 *J. Mod. Opt.* **55** 1
[17] Hasegawa A, Kodama Y 1995 *Solitons in optical communication*, Clarendon Press
[18] Caputo J-G, Maimistov A I 2002 *Physics Letters A* **296** (1) 34
[19] Schäfer T Wayne C E 2004 *Physica D* **196** 90