DYNAMIC PROJECTION OPERATOR METHOD IN THE THEORY OF HYPERBOLIC SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract: We consider a generalization of the projection operator method for the case of the Cauchy problem in 1D space for systems of evolution differential equations of first order with variable coefficients. It is supposed that the dependence of coefficients on the only variable x is weak, that is described by the introduction of a small parameter. Such problem corresponds, for example, to the case of wave propagation in a weakly inhomogeneous medium. As an example, we specify the problem to adiabatic acoustics in waveguides with a variable cross-section. Projection operators are constructed for the Cauchy problem to fix unidirectional modes. The method of successive approximations (perturbation theory) is developed and based on the pseudodifferential operators theory. The application of projection operators adapted for the case under consideration allows deriving approximate evolution equations corresponding to the separated directed waves.

Keywords: hyperbolic PDE, idempotents, inhomogeneous media, acoustics

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1. Introduction

The main idea of a division of the space of solutions of an evolution equation which corresponds to the so-called dispersion relations (that link frequency and the wave vector) goes back to the paper of Chu and Kovasznay [1]. The wave vector and therefore – the frequency are introduced via the Fourier transformation in space coordinates, which is effective almost exclusively in the case of a homogeneous background state (coefficients of equations independent on the coordinates). Links between dynamical variables also are obtained. A development of this idea is to combine the equations of the system under investigation in such a manner that would allow the evolution operator to be "diagonalized". Technically, both operations may be realized via application of a projection procedure [2]. More precisely, we use the idempotents built on the eigenvectors of the evolution operator [3–6]. The projectors solve both tasks: they combine the equations and change the dynamical variables. The idea of projection in a similar approach later has been also formulated in [7]. The presence of nonlinearity in a problem within such an approach has been realized in the spirit of the perturbation theory: the nonlinear terms have been combined by the same projection operators (built in the linearized theory) [8, 9], doing the step which has not been made in [1]. The only example of nonlinear non-perturbative corrections account has been realized by Riemann and, in the projection technique content in [10]. The general problems following the Riemann results (Riemann waves) have been investigated in the publications of Z. Peradzinski [11].

Briefly, the idea of this approach may be described by the following example. Consider an evolution problem as a system of two equations with constant coefficients.

$$\frac{\partial u(x,t)}{\partial t} - a\frac{\partial u(x,t)}{\partial x} - b\frac{\partial v(x,t)}{\partial x} = 0 \tag{1}$$

$$\frac{\partial v(x,t)}{\partial t} - c\frac{\partial u(x,t)}{\partial x} - d\frac{\partial v(x,t)}{\partial x} = 0 \tag{2}$$

A compact matrix form of (1)–(2)

$$\psi_t = L\psi \tag{3}$$

with

$$\psi = \begin{pmatrix} u \\ v \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} a\partial_x & b\partial_x \\ c\partial_x & d\partial_x \end{pmatrix} \tag{4}$$

introduces the evolution operator L and a state ψ of a system. The Fourier transformation in x

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{u}(k) e^{ikx} dk, \qquad v(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{v}(k) e^{ikx} dk \tag{5}$$

may be written as the matrix substitution

$$\psi = F\tilde{\psi} \tag{6}$$

Hence, in a compact notation of derivatives by an index, it yields a system of ordinary differential equations

$$\tilde{\psi}_t = F^{-1} L F \tilde{\psi} = \tilde{L}\tilde{\psi} \tag{7}$$

where, the k-representation of the evolution operator is

$$\tilde{L} = ik \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{8}$$

The matrix eigenvalue problem

$$\tilde{L}\phi = \lambda\phi \tag{9}$$

introduces two subspaces, which we would represent by the matrix of solutions Ψ

$$\tilde{L}\Psi = \Psi\Lambda \tag{10}$$

with diagonal matrix $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2\}$. We would choose the normalization of the eigenvectors such that the first component is a unit. It is easy to check that if $\lambda_1 \neq \lambda_2$, the inverse matrix exists and

$$\Psi^{-1}\tilde{L} = \Lambda\Psi^{-1} \tag{11}$$

Multiplying from the left side by Ψ^{-1} gives

$$L = \Psi\Lambda\Psi^{-1} \tag{12}$$

or, in components, it gives spectral decomposition of the matrix L

$$L_{ij} = \Psi_{ik}\Lambda_{kl}\Psi^{-1}_{lj} = \Psi_{ik}\lambda k\Psi^{-1}_{kj} = \sum_k \lambda_k \Psi_{ik}\Psi^{-1}_{kj} = \sum_k \lambda_k (P_k)_{ij} \tag{13}$$

Let us search for a matrix $\tilde{P}_i$, such that $\tilde{P}_i\Psi = \Psi_i$ should be eigenvectors of the evolution matrix in Eq. (10). Moreover, the standard properties of the orthogonal projection operators

$$\tilde{P}_i * \tilde{P}_j = 0, \qquad \tilde{P}_i^2 = \tilde{P}_i, \qquad \sum_i \tilde{P}_i = 1 \tag{14}$$

are implied. By means of (11) one can prove that

$$\tilde{P}_i = \Psi_i \otimes \Psi^{-1}_i \tag{15}$$

where Ψ_i is the i-th column and Ψ^{-1}_i – the i-th row of the corresponding matrix [12] and the identity

$$\tilde{L}\tilde{P}_i = \tilde{P}_i\tilde{L} \tag{16}$$

holds. The explicit form of the operators and variables in the mentioned normalization is given by

$$\Psi = \begin{pmatrix} 1 & 1 \\ \tilde{v}_1 & \tilde{v}_2 \end{pmatrix} \tag{17}$$

The values v_i are found from (10)

$$\tilde{v}_i = -\frac{i\lambda_i + ak}{bk} \tag{18}$$

$$\lambda_{1,2} = \frac{ik}{2}\left[(a+d) \pm \sqrt{(a-d)^2 + 4bc}\right] \tag{19}$$

if $\Delta = (a-d)^2 + 4bc > 0$, $\frac{\lambda_i}{i}$ are real and the equations are hyperbolic, which corresponds to the wave propagation, as Π and Λ are the right and left waves, respectively. Indeed,

$$\tilde{P}_1 = \frac{1}{\tilde{v}_2 - \tilde{v}_1}\begin{pmatrix} 1 \\ \tilde{v}_1 \end{pmatrix} \otimes (\tilde{v}_2, -1) \tag{20}$$

and

$$\tilde{P}_2 = \frac{1}{\tilde{v}_2 - \tilde{v}_1}\begin{pmatrix} 1 \\ \tilde{v}_2 \end{pmatrix} \otimes (-\tilde{v}_1, 1) \tag{21}$$

Let us go to new variables

$$\begin{aligned}\tilde{\Pi} &= (\tilde{P}_1\tilde{\psi})_1 = \frac{1}{\sqrt{\Delta}}\left(b\tilde{v}+\tilde{u}\left(\frac{1}{2}a-\frac{1}{2}d+\frac{1}{2}\sqrt{\Delta}\right)\right)\\ \tilde{\Lambda} &= \Lambda = \frac{1}{\sqrt{\Delta}}\left(\tilde{u}\left(\frac{1}{2}d-\frac{1}{2}a+\frac{1}{2}\sqrt{\Delta}\right)-b\tilde{v}\right)\end{aligned} \tag{22}$$

Its x-representation counterpart L and the inverse Fourier transforms of $\tilde{P}_i$

$$F\tilde{P}_iF^{-1} \equiv P_i \tag{23}$$

also commute; the proof is in the next line

$$F\tilde{L}F^{-1}F\tilde{P}_iF^{-1} = F\tilde{P}_iF^{-1}F\tilde{L}F^{-1} \tag{24}$$

Projecting the evolution (7) gives two independent equations, which are read as the first lines of

$$(P_i\psi)_t = LP_i\psi \tag{25}$$

the commutation (24) is taken into account. In the new variables

$$\Pi = (P_1\psi)_1, \qquad \Lambda = (P_2\psi)_1 \tag{26}$$

whence

$$P_1 = \frac{1}{v_2-v_1}\begin{pmatrix}1\\ v_1\end{pmatrix}\otimes(v_2,-1) \tag{27}$$

and

$$P_2 = \frac{1}{v_2-v_1}\begin{pmatrix}1\\ v_2\end{pmatrix}\otimes(-v_1,1) \tag{28}$$

are matrix operators that coincide in form with (20)–(21).

Applying the operator P_1 to the vector ψ yields

$$\Pi = (P_1\psi)_1 = \frac{1}{v_2-v_1}(v_2u-v) = \frac{b}{\lambda_2-\lambda_1}\left(\frac{\lambda_1-a}{b}u-v\right) \tag{29}$$

$$\Lambda = (P_2\psi)_1 = \frac{1}{v_2-v_1}(-v_1u+v) \tag{30}$$

It splits the original system into a system of independent equations directly from (25).

$$\begin{aligned}P_1L\psi = {} & \frac{1}{v_2-v_1}\begin{pmatrix}1\\ v_1\end{pmatrix}\otimes(v_2,-1)\begin{pmatrix}a\partial_x & b\partial_x\\ c\partial_x & d\partial_x\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix} = \\ & \frac{1}{v_2-v_1}\begin{pmatrix}1\\ v_1\end{pmatrix}\otimes(v_2,-1)\begin{pmatrix}a\partial_x u+b\partial_x v\\ c\partial_x u+d\partial_x v\end{pmatrix} = \\ & \begin{pmatrix}1\\ v_1\end{pmatrix}\otimes\frac{1}{v_2-v_1}\partial_x\left(v_2(au+bv)-(cu+dv)\right) = \\ & \begin{pmatrix}1\\ v_1\end{pmatrix}\otimes\frac{-1}{\sqrt{\Delta}}\left(\left(\left[a+d-\sqrt{\Delta}\right]\partial-2a\right)(au+bv)-2b(cu+dv)\right)\end{aligned} \tag{31}$$

$$\Pi_t = \frac{1}{2}\left(a+d+\sqrt{\Delta}\right)\Pi_x \qquad (32)$$
$$\Lambda_t = \frac{1}{2}\left(a+d-\sqrt{\Delta}\right)\Lambda_x$$

Let us take the case $a=d=0$ which is typical for physical applications in the wave theory (see also Sec. 6). Now $\lambda_{1,2} = \pm ik\sqrt{bc}$, $v_{1,2} = \pm ik\sqrt{\frac{c}{b}}$, therefore, (30) significantly simplifies

$$\Pi = \frac{1}{2}u + \frac{1}{2}\sqrt{\frac{b}{c}}v \qquad (33)$$

$$\Lambda = \frac{1}{2}u - \frac{1}{2}\sqrt{\frac{b}{c}}v \qquad (34)$$

$$\Pi_t = \sqrt{bc}\,\Pi_x, \qquad \Lambda_t = -\sqrt{bc}\,\Lambda_x \qquad (35)$$

such a system naturally describes propagation of the opposite one-dimensional waves (equivalent to the classic string equation) of acoustic [10] or electromagnetic waves [13] and many others, for example, for the electromagnetic environment with a division into right and left waves in [14].

The Cauchy problem has an elegant formulation in this context, e.g. for the system

$$\frac{\partial u(x,t)}{\partial t} - b\frac{\partial v(x,t)}{\partial x} = 0 \qquad (36)$$

$$\frac{\partial v(x,t)}{\partial t} - c\frac{\partial u(x,t)}{\partial x} = 0 \qquad (37)$$

with the initial conditions

$$u(x,0) = \phi(x), \qquad v(x,0) = \phi_1(x) \qquad (38)$$

It is directly reformulated for the system (35) for the mode variables

$$\Pi(x,0) = \left(P_1\psi(x,0)\right)_1 = \frac{1}{2}\phi(x) + \frac{1}{2}\sqrt{\frac{b}{c}}\phi_1(x) \qquad (39)$$
$$\Lambda(x,0) = \left(P_2\psi(x,0)\right)_1 = \frac{1}{2}\phi(x) - \frac{1}{2}\sqrt{\frac{b}{c}}\phi_1(x)$$

A D'Alembert-like formula follows directly from (35), solving the equations (35) by the characteristics method and applying the inverse formula

$$u = \Pi + \Lambda = \frac{1}{2}\phi\left(x+\sqrt{bc}t\right) + \frac{1}{2}\sqrt{\frac{b}{c}}\phi_1\left(x+\sqrt{bc}t\right) + \frac{1}{2}\phi\left(x-\sqrt{bc}t\right) - \frac{1}{2}\sqrt{\frac{b}{c}}\phi_1\left(x-\sqrt{bc}t\right) \qquad (40)$$

Note, that taking the elliptic case one concludes that it could be applied, for example, to a boundary problem of the Laplace/Poisson equation at half-plane.

In fact the formalism is not restricted by the case of two by two matrices and the dispersionless or non-dissipative case of the wave theory. It is effectively applied up to the 5×5 evolution operator of hydrodynamics [4] and recently to

4×4 evolution operator of electrodynamics [15]. It is developed for acoustics and plasma physics problems in the papers [10, 8, 16, 7] and [17], respectively. There are very interesting phenomena such as heating and streaming that appear in the interaction of the acoustic and zero frequency modes [10, 9]. Such theory may be considered also as a development of the Heaviside operator method as mentioned in [18, 19].

The challenge for a further development of this method is related to problems of evolution via differential operators with coefficients dependent on coordinates [18, 19]. Its main obstacle is in eventual simplifications after Fourier transformations. The only example that has been successfully solved relates to exponential stratification. The projection operator in this case has matrix elements which are integral operators with kernels defined via Hankel functions [20].

To analyze the solutions of the equations, it is useful to know in what physical conditions the equations were obtained. What is taken into account in the derivation of this equation? Traditionally, the wave fields are divided into components (entropic, acoustic, and vortical modes of [1], the last ones are subdivided into right and left waves). The eigenvectors of a linearized system are used to build the projection operators. The projection operators can select the relevant wave. Neither the Fourier transformation nor the dispersion relation can be effectively used for a general inhomogeneous medium.

The aim of this investigation is based on the ideas of the projection method, but it does not rely upon the Fourier transform. Nonetheless, we use its spirit in a form of pseudodifferential operators and the corresponding expansion. More exactly, we write the evolution operators directly (in the Fourier transform it is the parameter ω – frequency). In a solution subspace such operator is presented by a power series of an operator of the derivative in the basic space variable. Analogously, the matrix elements of the projection operators are built as similar expansions.

2. System of two equations with variable coefficients

Consider one-dimensional system of two equations with variable coefficients. It is a hyperbolic differential equation with variable coefficients in the domain, described in the introduction.

$$\frac{\partial u(x,t)}{\partial t} - a(x)\frac{\partial u(x,t)}{\partial x} - b(x)\frac{\partial v(x,t)}{\partial x} = 0 \tag{41}$$

$$\frac{\partial v(x,t)}{\partial t} - c(x)\frac{\partial u(x,t)}{\partial x} - d(x)\frac{\partial v(x,t)}{\partial x} = 0 \tag{42}$$

where a, b, c, d are coefficients of the evolution operator. It is implied further that the coordinate dependence is weak: the variations of the coefficients on a wavelength scale are supposed to be small. The Cauchy problem for (41) is specified by

$$u(x,0)=\phi(x), \qquad v(x,0)=\psi(x). \tag{43}$$

Applying the formal operator notations

$$\hat{a}(x)=a(x)D, \qquad \hat{b}(x)=b(x)D, \qquad \hat{c}(x)=c(x)D, \qquad \hat{d}(x)=d(x)D \tag{44}$$

we write a system:

$$\hat{a}(x)\tilde{u}(x)+\hat{b}(x)\tilde{v}(x)=\lambda(D)\tilde{u} \tag{45}$$

$$\hat{c}(x)\tilde{u}(x)+\hat{d}(x)\tilde{v}(x)=\lambda(D)\tilde{v} \tag{46}$$

that defines a pseudodifferential operator $\lambda(D)$.

Solving the system formally, $b \neq 0$, yields

$$\tilde{v}(x,t)=+\hat{b}(x)^{-1}\left(\lambda-\hat{a}(x)\right)\tilde{u} \tag{47}$$

This relation (47) may be considered as the link that defines the eigenvectors

$$\phi=\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} \tag{48}$$

for each $\lambda(D)$. Plugging the link (47) into (46) one obtains

$$\hat{c}(x)\tilde{u}(x)+\left(\hat{d}(x)-\lambda\right)\left[\hat{b}(x)^{-1}\left(\lambda-\hat{a}(x)\right)\tilde{u}\right]=0 \tag{49}$$

It gives an equation for the unknown operator λ:

$$\left\{-\lambda\hat{b}(x)^{-1}\lambda+\lambda\hat{b}(x)^{-1}\hat{a}(x)+\hat{d}(x)\hat{b}(x)^{-1}\lambda-\hat{d}(x)\hat{b}(x)^{-1}\hat{a}(x)+\hat{c}(x)\right\}\tilde{u}(x)=0 \tag{50}$$

3. Expansions and approximation

Suppose the operator $\lambda(D)$ is generally pseudo-differential, determined by the expansion

$$\lambda_i(D)=\sum_{n=0}^{\infty}s_n^{(i)}(x)D^n \tag{51}$$

Plugging (51) into (50) results in

$$\begin{aligned}\Bigg\{-\left(\sum_{n,m=-k}^{\infty}s_m^{(i)}(x)D^m\right)D^{-1}b^{-1}\left(s_n^{(i)}(x)D^n\right)+\\ \left(\sum_{n=-k}^{\infty}s_n^{(i)}(x)D^n\right)D^{-1}b^{-1}aD+\\ db^{-1}D^{-1}b^{-1}\left(\sum_{n=-k}^{\infty}s_n^{(i)}(x)D^n\right)-db^{-1}aD+cD\Bigg\}\tilde{u}(x)=0\end{aligned} \tag{52}$$

For each mode, defined on a space S, such that the series are asymptotic, having in mind a possibility to use a finite number of terms on a subspace $S_i \in S$. It corresponds to the so-called short wave approximation in many works devoted to the wave propagation theory [4].

Restricting ourselves by the three-term approximation $\lambda=p+qD+rD^2$, we get:

$$\begin{gathered}\Big\{-\left(p+qD+rD^2\right)D^{-1}b^{-1}\left(p+qD+rD^2\right)+\left(p+qD+rD^2\right)D^{-1}b^{-1}aD+\\ db^{-1}\left(p+qD+rD^2\right)-db^{-1}aD+cD\Big\}\tilde{u}(x)=0\end{gathered} \tag{53}$$

Next, suppose that the derivatives of the original system (41) coefficients are of the minor order compared with the coefficients themselves, having in mind the mentioned supposition of short waves (or slow varied coefficients.) Then, equalizing the coefficients by powers of D, taking the chosen order of derivatives into account

$$\begin{aligned} &D^0\colon -pb^{-1}q+p\left(b^{-1}r\right)'-qb^{-1}p+pb^{-1}a-r\left(b^{-1}p\right)'+db^{-1}a=0\\ &D^1\colon -pb^{-1}r-qb^{-1}q+rb^{-1}p-r\left(b^{-1}q\right)'+qb^{-1}a+r\left(b^{-1}a\right)'+\\ &\qquad db^{-1}q+c-db^{-1}q=0\\ &D^2\colon -qb^{-1}r-rb^{-1}q-r\left(b^{-1}r\right)'+rb^{-1}a+db^{-1}r=0\end{aligned} \tag{54}$$

The commutation relations are used in the transformations.

Multiplying the second equation on $b\neq 0$ we get two branches of the operator $\lambda(D)$:

$$\begin{gathered} p=0, \qquad r=0\\ q_\pm=\frac{(a+d)\pm\sqrt{(a-d)^2+4bc}}{2}\end{gathered} \tag{55}$$

that define two modes of the solution. The relations coincide with those for constant coefficients from the introduction in the order under consideration

$$\lambda_\pm=q_\pm D \tag{56}$$

that supports the result.

4. Projection operators

Going to a generalization of the method described in the introduction let us consider a 2×2 matrix with operator-valued non-commuting elements

$$P=\begin{pmatrix} p & \pi\\ \xi & \eta\end{pmatrix} \tag{57}$$

with the basic determining idempotent condition

$$P^2=P \tag{58}$$

It immediately yields

$$P=\begin{pmatrix} p & \pi\\ \pi^{-1}\left(p-p^2\right) & 1-\pi^{-1}p\pi\end{pmatrix} \tag{59}$$

There are possibilities of the choice of the operators p, π that fix the projection subspaces (48)

$$\begin{pmatrix} p & \pi\\ \pi^{-1}\left(p-p^2\right) & 1-\pi^{-1}p\pi\end{pmatrix}\begin{pmatrix} u\\ v\end{pmatrix}=\begin{pmatrix}\tilde{u}\\ \tilde{v}\end{pmatrix} \tag{60}$$

Using this equality and the condition of completeness $P_+ + P_- = I$, we obtain the explicit form of the projection operators that correspond to the two versions of λ for both $q_\pm$ given by (55).

5. Particular case

We consider a more compact, still hyperbolic case $a=0, d=0$, $bc>0$:

$$\frac{\partial u(x,t)}{\partial t} - b(x)\frac{\partial v(x,t)}{\partial x} = 0 \tag{61}$$

$$\frac{\partial v(x,t)}{\partial t} - c(x)\frac{\partial u(x,t)}{\partial x} = 0 \tag{62}$$

related to physical problems, as mentioned. The projection operators in this case are calculated via the definition (57) for the projection subspaces (48):

$$P_{1,2} = \frac{1}{2}\begin{pmatrix} 1 & \pm\left(f - D^{-1}f'\right)^{-1} \\ \pm\left(f - D^{-1}f'\right) & 1 \end{pmatrix} \tag{63}$$

where

$$f = \sqrt{\frac{c(x)}{b(x)}} \tag{64}$$

f' denotes the derivative of f. Now the evolution operator L (see, for example, (4)), in the same notations, simplifies:

$$L = \begin{pmatrix} 0 & b(x)D \\ c(x)D & 0 \end{pmatrix} \tag{65}$$

The commutator of L and P_1 is equal to

$$[P_1, L] = \begin{pmatrix} D^{-1}f^{-1}DcD - bfD & 0 \\ 0 & D^{-1}fDbD - cf^{-1}D \end{pmatrix} \tag{66}$$

because the identities $f - D^{-1}f' = D^{-1}fD$ and $\left(f - D^{-1}f'\right)^{-1} = D^{-1}f^{-1}D$ hold.

The condition that the commutator is zero can be written as

$$D^{-1}f'bf = 0 \tag{67}$$

or with the explicit expression for f (64):

$$\frac{1}{2}D^{-1}\left(c' - \frac{b'}{b}c\right) = 0 \tag{68}$$

It fixes the case of a complete reduction (diagonalisation) of the evolution operator.

As a further development of the method we suggest an approximate procedure (see *e.g.* [21]) Using the projection operators we shall found new equations for left and right waves, splitting the problem of evolution. The approximate splitting is achieved, if the commutators of $P_{1,2}$ and L could be neglected. It is possible, if the coefficients b, c are of the zero order ($\cong O(1)$), while the order of the derivative $\left(\frac{c}{b}\right)'$ is of a higher order, e.g. $\cong O(\epsilon)$ with ϵ as the mentioned parameter of the inhomogeneity. Acting by the projection operator P_1 to the system (41)

$$P_{1,2}\Psi_t = P_{1,2}L\Psi \tag{69}$$

or, approximately

$$(P_{1,2}\Psi)_t = L(P_{1,2}\Psi) \tag{70}$$

The result of the operation gives a possibility to introduce notations of the mode variables Π, Λ via

$$P_1\Psi = \frac{1}{2}\begin{pmatrix} 1 & D^{-1}f^{-1}D \\ D^{-1}fD & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \Pi \\ (f - D^{-1}f')\Pi \end{pmatrix} \tag{71}$$

$$P_2\Psi = \frac{1}{2}\begin{pmatrix} 1 & -D^{-1}f^{-1}D \\ -D^{-1}fD & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \Lambda \\ -D^{-1}fD\Lambda \end{pmatrix} \tag{72}$$

Reading the first lines of the relations yields

$$\Pi = \frac{1}{2}\left(u + D^{-1}f^{-1}Dv\right) \tag{73}$$

and

$$\Lambda = \frac{1}{2}\left(u - D^{-1}f^{-1}Dv\right) \tag{74}$$

that gives explicit expressions for the mode variables. From these expressions it follows that:

$$\begin{aligned} u &= \Pi + \Lambda, \\ v &= \left(f - D^{-1}f'\right)(\Pi - \Lambda) \end{aligned} \tag{75}$$

These relations allow stating the Cauchy problems for directed waves.

Considering the equations (52), (56) and an approximate relation for the commutator $P_1L = LP_1 - [P_1, L]$ one obtains the evolution equations of the modes:

$$\Pi_t = \sqrt{bc}\,\Pi_x \tag{76}$$

$$\Lambda_t = -\sqrt{bc}\,\Lambda_x \tag{77}$$

Solving the first order equations by the method of characteristics gives the well-known results, but velocity is a function depending on the coordinate.

6. Example of acoustic waves

Let the basic fluid variable, the pressure, density and velocity be denoted as p, ρ, $\vec{v}$ respectively. The momentum (Euler) equation for a compressible fluid is

$$\rho\frac{\partial\vec{v}}{\partial t} = -\nabla p + f \tag{78}$$

The continuity equation reads

$$\frac{\partial p}{\partial t} + \nabla(\rho\vec{v}) = 0 \tag{79}$$

that, together with the energy equation and the equations of state, closes a dynamic problem for the fluid.

Consider a linearization in a one-dimensional case with respect to the perturbations marked by the primes $\rho = \rho_0 + \rho'$, $p = p_0 + p'$, $v = v'$. The unperturbed (ground state) pressure p_0 and density ρ_0 variables are supposed to be dependent on the space variable x. The system for the perturbations reads as

$$\rho_0 \frac{\partial v'}{\partial t} = -\frac{\partial p'}{\partial x} \tag{80}$$

$$\frac{\partial \rho'}{\partial t} = \rho_0 \frac{\partial v'}{\partial x} + v' \frac{\partial \rho_0}{\partial x} \tag{81}$$

It is known that such case without dissipation leads to the adiabatic condition:

$$\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma} \tag{82}$$

Where γ is the heat capacity ratio. Its account leads to the system:

$$\left(\rho_0 v'\right)_t + \frac{p_0}{\rho_0}\gamma \rho'_x = 0 \tag{83}$$

$$\rho'_t - \left(\rho_0 v'\right)_x = 0 \tag{84}$$

The notations $u = \rho'$, $\rho_0 v' = v$ and $b = 1$, $\frac{p_0}{\rho_0}\gamma = c(x)$ establish the correspondence of this system with the system (61)–(62). In this case where the variable f is expressed as $f = \sqrt{\frac{p_0}{\rho_0}\gamma}$ that now coincides with $\sqrt{bc}$ we can name a local velocity propagation of the acoustic wave. In the waveguide case the dependence of the transversal mode velocity on the longitude coordinate may be caused by the waveguide dimension. Hence, the combination of the pressure and velocity perturbations as in the relations for the projection operators that define the right and left waves (73)–(74) solves the problem of such wave initialization. The evolution in the weakly inhomogeneous medium in the first approximation is solved by a characteristic method mentioned in connection with (76)–(77).

7. Conclusion

The development of the method of dynamic projection operators for the theory of hyperbolic systems of partial differential equations with variable coefficients was considered. This idea has been presented in [22]. The result allows accounting for a nonlinearity to be introduced as a perturbation by the amplitude parameter, and application of the projection operators leads to the interaction of modes.

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