ON PROBLEM OF ROSSBY AND POINCARE ATMOSPHERIC WAVES SEPARATION AND SPECTRA

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Abstract: We propose a new inverse problem formulation based on the hydrodynamics consideration of a gas/water fluid that results in planetary waves diagnostics. We analyze such a possibility beginning from a simplest version of geophysical hydrodynamics, written in the β-plane model. The problem of diagnostics is solved approximately after expansion with respect to the transverse basis functions applying projecting to Rossby and Poincare waves in each transverse subspace that contains its superposition. The corresponding discrete version of the operators is built to be applied to the observation data.

1. Introduction

One of the important problems of geophysical hydrodynamics is a reliable identification of wave disturbances directly on the basis of observations, especially in the case when the data represents a mixture of different wave types of similar scales or periods. Such knowledge is necessary for formulating a forecast problem as well as describing the medium parameters predicting its variations.

The conventional methods of the existing data processing do not allow solving such a problem with the necessary precision. Namely, the method uses harmonic (spectral) analysis of time series observations. The success of their application is determined by the availability of sets of experimental data obtained in long-term (in comparison with the wave period) atmospheric observations at a large number of independent observation stations. The existing system of observations in the atmosphere, both terrestrial and satellite, covers the Earth's surface extremely unevenly and, despite the length of observations, does not

allow solving the problem of wave identification. In addition, as a rule, only certain atmospheric parameters are determined in observations, for example, the wind speed or the temperature in a limited range of heights only, over a space that is limited by latitude and longitude. Thus, the problem of identifying wave disturbances faces fundamental difficulties.

Significant progress in its solution involves the use of fundamentally new methods for analyzing observations. Among these methods is the method of projection operators. In the method of projection operators it is assumed that the observed spatio-temporal structure of the atmosphere is determined by the superposition of waves of different types. For example, the variations in atmospheric parameters with time scales of several hours may be due to the propagation of acoustic and internal gravitational waves of appropriate scales, and variations with periods exceeding 24 hours are caused by planetary waves. For each type of waves participating in such a superposition, dispersion and polarization relations (the relation between the wave-vector components) are assumed to be known. On the basis of such assumptions, it is possible to construct projection operators for the initial superposition state Φ on a linear basis corresponding to a known type of atmospheric waves

$$\Phi = \sum_1^n \Phi_i = \sum_1^n P_i \Phi \tag{1}$$

Here P_i and Φ_i are the projection operator and the wave vector, respectively corresponding to the i-th type of the wave. The vector $\Phi_i = P_i \Phi$ contains components of the wave field, for example, the meridional and zonal projection of the velocity vector, pressure, etc. The relation between the components of the vector Φ_i for each type of wave is determined by the corresponding eigenvalue. The action of the projection operator on the superposition state Φ, which, in our context, which is considered as a result of observations, determines the amplitudes and phases of waves of a known type.

In the first section, the basic system of equations for barotropic planetary waves is presented. The expansion by the eigen functions of the transversal operator reduces the evolution to one-dimensional that, by means of a standard procedure, is solved and leads to construction of dynamic projecting operators for a given transversal mode. The second section is devoted to details of the projecting technique features adjusted to the planetary wave in the upper atmosphere. The corresponding expansion of the projecting operators in a Taylor series with respect to frequency is introduced. The next section presents discretization of the operators to be applied to the observation data.

2. Projecting operators for planetary waves

Let us consider a problem of construction of the projecting operators for barotropic Rossby and Poincare waves. Such theory has been considered in [1, 2]

but needs reformulation for the boundary problem outlined in the introduction. Starting from the basic system [3]

$$U_t - fV + c^2\eta_x = 0$$
$$V_t + fU + c^2\eta_y = 0 \qquad (2)$$
$$\eta_t + V_y + U_x - \beta V = 0$$

In (2) the variables x, y define the zonal (eastward) and meridional (southward) directions, t – time. The variables U and V are connected with the meridional (u) and zonal (v) velocity components: η – geopotential height, H_0 is the height of a homogeneous atmosphere, $vH_0 = V$, $uH_0 = Uf = 2\Omega\cos(\theta)$, $H = H_0(1-\beta y)$, $c^2 = gH$, $\beta = 1/R\sin(\theta)$, R is the radius of the Earth, f – the angular velocity of the Earth's rotation, β – the colatitude on which the plane is defined. Equations (2) describe the dynamics of gas, analogous to the motion in a channel of width L with impermeable walls and the bottom slope determined by the parameter β. The solution for this case is represented in the following form [1, 2]:

$$V = \sum_n Y_n \Theta_n$$
$$U = \sum_n Y_n \varphi_n + Y_{ny}\phi_n \qquad (3)$$
$$\eta = \sum_n Y_n \mu_n + Y_{ny}\nu_n$$

where the basic functions and the Fourier transformation are presented as

$$Y_n = \sin(l_n y)\exp(\beta y/2) \qquad l_n = \pi n/L$$
$$\Theta = \int \exp(ikx)\tilde{\theta}dk \qquad (4)$$

Plugging it into (3) yields

$$\tilde{\mu}_t + ik\tilde{\varphi} - \beta\tilde{\theta} = 0$$
$$\tilde{\varphi}_t - f\tilde{\theta} + ikc^2\tilde{\mu} = 0 \qquad (5)$$
$$\tilde{\theta}_t + f(1-Q)\varphi + \beta c^2 Q\tilde{\mu} = 0$$
$$Q = (l^2 + \beta^2/4)/(\beta^2 - f^2/c^2)$$

Forming a vector

$$\Phi^T = \left\{\tilde{\mu}, \tilde{\varphi}, \tilde{\theta}\right\} \qquad (6)$$

one has the evolution of Fourier components in a matrix form

$$\Phi_t + L(ik)\Phi = 0 \qquad (7)$$

The condition for the solvability of the system (4) is the dispersion equation [3, 1, 2] the roots of which give relations for the low-frequency component (Rossby waves), approximately

$$\sigma_1 = -\beta fk/(k^2 + l^2 + f^2/c^2) \qquad (8)$$

in the range

$$\sigma_1^2/f^2 < 1 \tag{9}$$

and the high-frequency component (Poincare waves) that has the form:

$$\sigma_{2,3} = \pm c((f/c)^2 + k^2 + l^2 + \beta^2/4)^{1/2} \tag{10}$$

in conditions of $\sigma_{2,3}/f > 1$. Here, σ_1 is the Rossby wave frequency, and $\sigma_{2,3}$ are approximate values of the right and left Poincare waves; k and l are the components of the wave vector in the zonal and meridional directions, respectively. Generally

$$\Phi = \sum_1^3 \Phi_i \tag{11}$$

The relationships of the vector components for the waves under consideration (polarization relations) are determined as follows

$$\begin{aligned} \tilde{\varphi}_i &= \frac{f\sigma_i + k\beta c^2}{\beta\sigma_i + kf}\tilde{\mu}_i \equiv a_i\tilde{\mu}_i \\ \tilde{\theta}_i &= \frac{\sigma_i^2 - k^2 c^2}{i(\beta\sigma_i + kf)}\tilde{\mu}_i \equiv b_i\tilde{\mu}_i \end{aligned} \tag{12}$$

Hence, the vectors can be written as

$$\Phi_i = \begin{pmatrix} 1 \\ a_i \\ b_i \end{pmatrix} \tilde{\mu}_i \tag{13}$$

For barotropic Rossby and Poincare waves in the atmosphere, the general form of the projection operators was obtained in [1, 2] and has the form

$$P_i = \begin{pmatrix} \alpha_i & \beta_i & \gamma_i \\ \alpha_i a_i(\sigma) & \beta_i a_i(\sigma) & \gamma_i a_i(\sigma) \\ \alpha_i b_i(\sigma) & \beta_i b_i(\sigma) & \gamma_i b_i(\sigma) \end{pmatrix} \tag{14}$$

where the parameters α_i are expressed via

$$\alpha_i = \frac{\Delta_i}{\Xi} \tag{15}$$

with

$$\Delta_1 = b_2 a_3 - b_3 a_2 \qquad \Delta_2 = b_3 a_1 - b_1 a_3 \qquad \Delta_3 = b_1 a_2 - b_2 a_1 \tag{16}$$

and, next

$$\Xi = \sum_i \Delta_i \tag{17}$$

The parameters β_i are expressed by

$$\beta_1 = \frac{(b_3 - b_2)}{\Xi} \qquad \beta_2 = \frac{(b_1 - b_3)}{\Xi} \qquad \beta_3 = \frac{(b_2 - b_1)}{\Xi} \tag{18}$$

and, finally, γ_i are:

$$\gamma_1 = \frac{(a_2 - a_3)}{\Xi} \qquad \gamma_2 = \frac{(a_3 - a_1)}{\Xi} \qquad \gamma_3 = \frac{(a_1 - a_2)}{\Xi} \tag{19}$$

The conventional properties of the projecting operators hold

$$[P_i]^2 = P_i \tag{20}$$

with the completeness condition

$$\sum_i P_i = I \tag{21}$$

We also note that the solution of the problem of separating a particular wave from a superposition assumes the establishment of parameters such as amplitude, frequency, and wave number. Indeed, with the direct application of projection operators, only the amplitude of the wave (1) is determined. Thus, the method of projection operators determines the procedure for extracting the characteristics of a known planetary wave (amplitude, frequency and wave number) from the superposition state.

3. Features of application of projection operators for planetary waves in the upper atmosphere

The application of the method of projection operators to analyze the wave structure of the upper atmosphere has certain features, determined both by the medium itself and by the nature of the experimental data obtained in the observations. The fact is that in most cases the identification of waves in the atmosphere is carried out as a result of the analysis of time series of observations performed at individual stations. Therefore, the design operators obtained earlier should be modified, defining their action only on time-dependent functions (a series of observations). To this end, it is necessary to determine the dependence from the dispersion relations and use it in the expressions for the matrix elements of the projection operators. Consider such transformations for selected types of planetary waves. The dispersion relation (8) for Rossby waves can be represented in the form of an algebraic equation determining the dependence $k = k(\sigma)$:

$$k^2 + k_{\max}^2 - 2\sigma_{\max} k_{\max} k / \sigma = 0 \tag{22}$$

Here $k_{\max}^2 = f^2/c^2 + l^2$ and $\sigma_{\max} = -\beta f / 2k_{\max}$. The solutions of this equation for the wave number k determine "long" (and "short") Rossby waves corresponding to a single frequency. The parameter $k_{\max}$ is the wave number corresponding to the Rossby wave with the maximum permissible frequency $\sigma_{\max}$. It is clear that $\sigma_{\max}$ (or the minimum period corresponding to this value) depends on the latitude. Figure 1 shows a graph of the minimum period of Rossby waves, depending on the colatitude.

$$k_{\max} = \sqrt{(f/c)^2 + l^2} \tag{23}$$

We note that in Figure 1 the line corresponding to $l = 0$ determines the minimum Rossby wave period, which is essentially unreachable, since the solution with such a projection of the wave number is absent in (3). An estimate of the
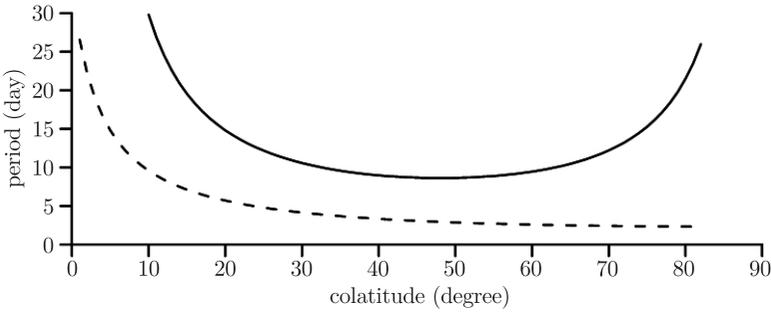
Figure 1. The minimum periods of the Rossby wave are $l=0$ (dashed line) and $l=1$ (solid line), depending on the colatitude; the y-axis is the periods (day), the x-axis is the colatitude (degrees)
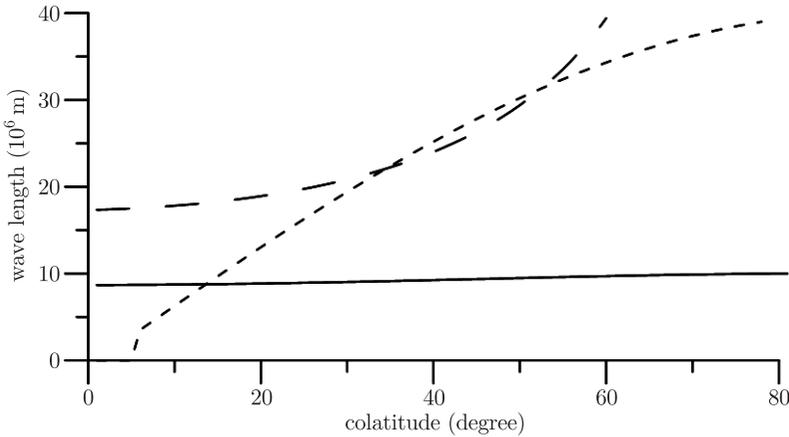
Figure 2. The Rossby wavelength at $l=0$ (solid line), $l=1$ (dash-dotted line), corresponding to the minimum period, and the length of the circle in the Earth's atmosphere (dashed line); the y-axis is the wavelength (thousand km); the x-axis is the colatitude (degrees)

Rossby wave period, which has a meridional structure, was carried out on the assumption that

$$l = \pi/L \qquad L = \pi R/2 \tag{24}$$

Figure 1 shows that the minimum possible periods of barotropic Rossby waves at medium latitudes exceed 4 days.

According to experimental observations, perturbations with periods exceeding the minimum Rossby wave period are fairly well recorded in the upper atmosphere. Figure 2 shows the Rossby wavelengths corresponding to the maximum frequencies (minimum periods) for different values of the meridional projection of the wave vector, depending on the colatitude, and also the latitude circle length on the Earth's surface. As can be seen from Figure 2, the Earth's dimensions allow the realization of both "long" and "short" Rossby waves at latitudes less than 70° in the upper atmosphere. Given the possibility of such implementation, it is necessary to consider two sets of projection operators corresponding to "long" and

"short" Rossby waves. The dependence $k(\sigma)$ of Taylor expansion upto the terms of the order up to

$$\left(\frac{\sigma}{\sigma_{\max}}\right)^3 \tag{25}$$

has the form (26) for "short" waves

$$k = 2\frac{k_{\max}\sigma_{\max}}{\sigma} - \frac{k_{\max}\sigma}{2\sigma_{\max}} \tag{26}$$

and (27) for "long waves"

$$k = \frac{1}{2}k_{\max}\frac{\sigma}{\sigma_{\max}} + \frac{1}{8}k_{\max}\frac{\sigma^3}{\sigma_{\max}^3} \tag{27}$$

The coefficients in the operator (14) $a_1(\sigma)$, $b_1(\sigma)$ in the "short" wave approximation will have the form (28)–(29),

$$\alpha_1(\sigma) = c\left(\gamma + \frac{(\gamma^2 - 1)(\sigma/f)^2}{\gamma}\right) \tag{28}$$

$$b_1(\sigma) = \frac{c}{i}\left(\frac{\gamma f}{\sigma} + \frac{(\gamma^2 - 1)\sigma}{\gamma f}\right) \tag{29}$$

and for "long" waves the form (30)–(31):

$$a_1(\sigma) = c\frac{(4q^2 - \gamma^2)}{\gamma(4q^2 - 1)} + \frac{(1 - \gamma^2)\sigma^2}{(4q^2 - 1)^2\gamma f^2} \tag{30}$$

$$b_1(\sigma) = \frac{c(16q^4 - \gamma^2)\sigma}{4f\gamma iq^2(4q^2 - 1)}. \tag{31}$$

with $i = \sqrt{-1}$

$$\gamma = \beta\frac{c}{f} \tag{32}$$

and

$$q = \frac{\sigma_{\max}}{f} \tag{33}$$

For Poincare waves, the projection operator is constructed in the same way. The dispersion relation for these waves has the form:

$$c^2k^2 = \sigma^2 - \sigma_0^2 \tag{34}$$

and

$$\sigma_0^2 = f^2\left(1 + \frac{\gamma^2}{4}\right) + c^2l^2 \tag{35}$$

It follows from the dispersion relation (35) that ω_0 is the minimum frequency of Poincare waves and its solution with respect to the wave number determines two waves propagating eastward (Index 2) and westward (Index-3). The dependence of Poincare waves on frequency is determined from (35) by the following expressions:

$$k_{2,3}(\sigma) = \pm\frac{\sigma}{c}\left(1 - \frac{\sigma_0^2}{2\sigma^2}\right) \tag{36}$$

Unlike the operator for Rossby waves $y = f/\sigma$ is chosen as the small parameter for expansion in the Poincare wave operator. To simplify the further procedure

for calculating the matrix elements of the operators design, it is convenient to consider the dependence on the latitude of the characteristic parameters for the types of waves under consideration. Obviously, the values of these parameters determine the limiting amplitude values of small parameter changes for Rossby waves and Poincare waves. The dependence of these parameters on the colattitude is shown in Figure 3. As can be seen from Figure 3, at midlatitudes the values of small parameters x and y are less than 0.5, which should ensure good convergence of the power series in expansions in these parameters.
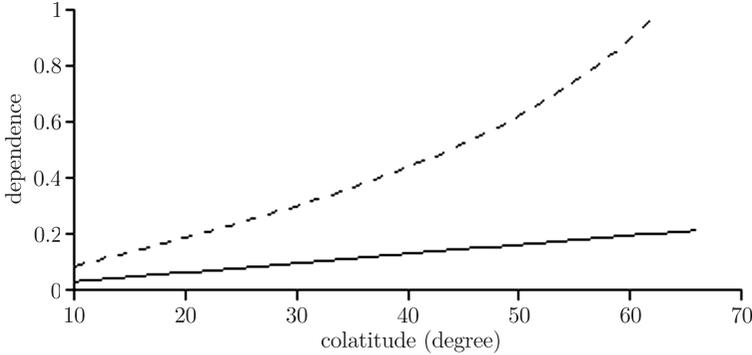
Figure 3. Dependence of parameters $q = \frac{\sigma_{\max}}{f}$ (solid line) and $\gamma = \frac{\beta c}{f}$ (dased line) on colatitude

Neglecting the small terms of order q^2 in (12), we can determine the following expressions for evaluation of the coefficients $a_{2,3}(\sigma)$ and $b_{2,3}(\sigma)$:

$$a_2(\sigma) = c - \frac{1}{8}c(\gamma-1)\frac{\gamma^2}{q^2(\gamma+1)}y^2 \quad b_2(\sigma) = \frac{1}{4}\gamma^2\frac{1}{q^2}\frac{c}{i(\gamma+1)}y \tag{37}$$

$$a_3(\sigma) = -c + \frac{c}{8}\frac{(\gamma+1)\gamma^2}{q^2(\gamma-1)}y^2 \qquad b_3(\sigma) = \frac{1}{4}\frac{\gamma^2}{q^2}\frac{c}{i(\gamma-1)}y \tag{38}$$

Values for the coefficients α^i, β^i, γ^i can be obtained from the expressions (15), (28), (29), (37), (38). It is convenient to represent matrix elements in the projection operator in the form of an expansion in the power series in the small parameter $x = \sigma/f$ for Rossby waves

$$P^1_{ij} = \sum A^1_{ijn}x^n$$

$$P^{2,3}_{ij} = \sum A^{2,3}_{ijn}y^n \tag{39}$$

in parameter $y = f/\sigma$ – for Poincare waves.

The necessary order in the expansion can be established by determining the accuracy of the normalization conditions for the projection operators. A direct calculation of the matrix elements in the projection operator showed that to satisfy these conditions with an accuracy of ~ 10 percent, it is necessary to take into account the terms of the expansion of the third order of smallness for

Rossby waves and the second order of smallness for Poincare waves. Thus, after performing the transformations (15)–(38), the matrix elements of the operators will be represented in the form of power series (44) over a small parameter (x for Rossby waves and y for Poincare waves). At this stage, pre-test operations of the constructed design statements can be performed. We use condition (1) in the form:

$$P^i \Phi = \Phi_i \qquad (40)$$

We apply (40) to test the constructed projection operator, assuming that it is an arbitrary wave from the considered class of planetary waves. In this case, the use of the projection operator for Rossby or Poincare waves should give the amplitude of the wave corresponding to it. The amplitudes of the other waves entering the superposition must go to zero.
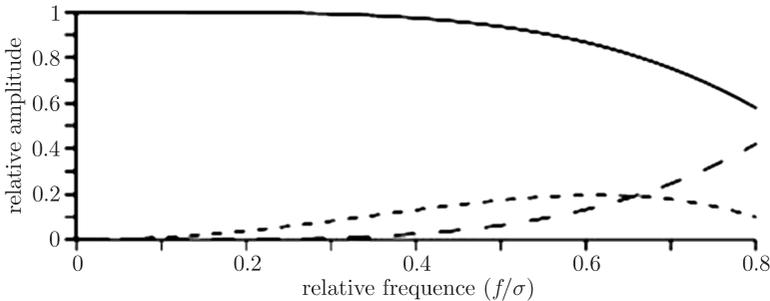
Figure 4. Amplitudes depend on frequency ($x = f/\sigma$) for Poincare wave which propagates eastward after applying the projection operator to the wave field containing Poincare wave propagating eastward (solid line), the Poincare West wave (dotted line), and the Rossby wave (dashed line)

In Figure 4 the application of the projection operator for Poincare waves propagating eastward to a superposition wave field containing a Poincare wave propagating eastward, a long Rossby wave, and a Poincare wave propagating westward is shown. As can be seen from Figure 5, the projection operator uniquely divides the Poincare waves propagating in different directions at different frequencies. At the same time, the "short" Rossby wave becomes comparable in amplitude with the released Poincare wave at frequencies of ~ 0.5. Given the range of changes in the parameter y, (see Figure 3), we can assume that the constructed operator solves the problem of separating the Poincare wave in the frequency region corresponding to the condition $y < 0.5$. We also note that the wave amplitudes obtained in (40) are instantaneous and, therefore, taking into account the essential frequency difference for the Rossby and Poincare waves, it is possible to expand the scope of the constructed operator, analyze the time evolution of the amplitudes of the emitted waves. Similar testing for other design operators yields similar results in Figure 5.

Thus, the constructed projection operators, which depend only on frequency, allow solving the problem of determining the type of wave and its charac-
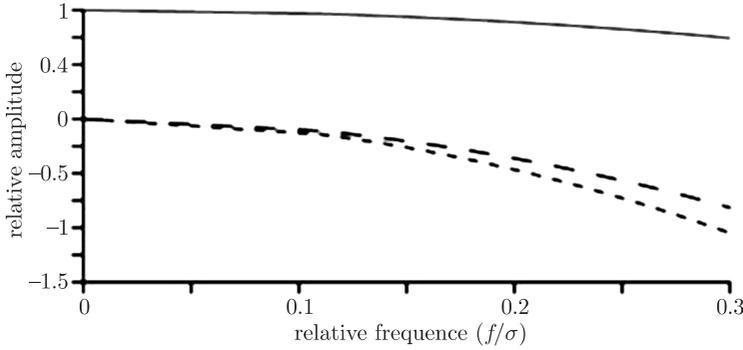
Figure 5. Amplitudes as function of frequency ($x = \sigma/f$); short Rossby waves after applying
the projection operator to the wave field containing the Rossby wave (solid line)
and the Poincare wave (dotted and dashed lines); at the same time, the "long" Rossby wave
becomes comparable

teristics-amplitude, frequency, wave numbers, based on the analysis of the initial
wave field-experimental data obtained in observations. Similar testing for other
design operators yields similar results in Figure 5.

4. Finite-difference analogues of projection operators

For direct application of projection operators to experimental data, it is
necessary to redefine the frequency operator. According to the initial assumptions,
the constructed operators act on the waves of the form for

$$f(\vec{r},t) = f_0 \exp\big(i(\vec{k}\vec{r} - \sigma t)\big) \tag{41}$$

This allows us to associate the multiplication with the frequency with the time
differentiation operator, that is, division with integration

$$\sigma f(\vec{r},t) \rightarrow i\frac{\partial f(\vec{r},t)}{\partial t} \tag{42}$$

$$\sigma^{-1} f(\vec{r},t) \rightarrow -i\int f(\vec{r},t)dt \tag{43}$$

In view of this, the variables xn and yn in the projection operators in the
transition from the (σ,k)-space to the (r,t)-space must be transformed into the
corresponding operators of differentiation and time integration of order n. Thus,
using the transformations (43), it is possible to construct finite-difference pro-
jection operators whose action on a time-varying wave field will determine the
characteristics of the corresponding planetary wave. The main task in the im-
plementation of the whole procedure is to reduce the errors in the numerical
approximations of the differential and integral operators entering into the projec-
tion operator. In order to reduce the numerical errors, it is convenient to go from
the constructed integro-differential projection operators to operators containing

only derivatives with respect to time. In general form, the constructed projection operators (44) can be rewritten in the form: (σ, k)

$$
\begin{aligned}
P^1 &= A^1 f/\sigma + B^1 + C^1 \sigma/f + D^1 \sigma^2/f^2, \\
P^{2,3} &= A^{2,3} \sigma/f + B^{2,3} + C^{2,3} f/\sigma + D^{2,3} f^2/\sigma^2
\end{aligned}
\tag{44}
$$

For the Rossby waves and for the Poincare waves, respectively, A^i, B^i, C^i, D^i – matrices of coefficients in projection operators, f – is the Coriolis parameter. For the constructed projection operators, the commutativity property with the differentiation operator is valid. In this case, elementary transformations make it possible to pass from the integro-differential operators (23) to differential operators of the matrix coefficients:

$$
A^i, \; B^i, \; C^i, \; D^i
\tag{45}
$$

hence

$$
\begin{aligned}
\sigma P^1 &= A^1 f + B^1 \sigma + C^1 \sigma^2/f + D^1 \sigma^3/f^2 \\
\sigma^2 P^{2,3} &= A^{2,3} \sigma^3/f + B^{2,3} \sigma^2 + C^{2,3} f \sigma + D^{2,3} f^2
\end{aligned}
\tag{46}
$$

Further, considering that, we assume that the operator determines the action of the projection operator on the time-differentiated wave field under study.

$$
P^i \sigma = \sigma P^i
\tag{47}
$$

Thus, the proposed procedure, without loss of generality. allows us to apply the constructed operators to the wave field analysis. Derivatives with respect to time from the initial wave field can be represented by finite-difference operators of the form:

$$
\begin{aligned}
\sigma P^i &= i P^i \frac{\partial}{\partial t} \\
\sigma^2 P^i &= -P^i \frac{\partial^2}{\partial t^2} \\
\sigma^3 P^i &= -i P^i \frac{\partial^3}{\partial t^3}
\end{aligned}
\tag{48}
$$

A time-varying wavefield was determined in accordance with the polarization relations (12) with the use of the mentioned approximations (48). A test calculations of the projection operator's action on the initial wave field for a given frequency from the possible planetary waves amplitude range may be realized by means of transition to discrete aproximation as below in (49)–(50):

$$
\frac{\partial f}{\partial t} = \frac{f(t) - f(t-\tau)}{\tau}
\tag{49}
$$

$$
\frac{\partial^2 f}{\partial t^2} = \frac{f(t) - 2f(t-\tau) + f(t-2\tau)}{\tau^2}
\tag{50}
$$

The projection operator for a given type of wave was applied to the wave field so constructed. In this case, one can compare the characteristics of the wave, selected after the application of the projection operator with the original harmonic wave. The results of calculations showed that in those cases when the type of wave given in the initial wave field coincided with the type of the projection operator,

the amplitude of the wave was determined with an accuracy of not less than 5 percent, otherwise, when the projection operator was applied not to its wave, the amplitude of the wave determined by the projection operator was less than the amplitude of the original wave by not less than 4–5 times.

5. Conclusion

In this paper in the expressions for matrix elements of the projecting operators we used expansions for the wave numbers $k_i(\sigma)$ in a series by frequency for each branch of the dispersion relation. It is possible for a specific frequency range and allows expressing the projectors in terms of a differential operator (derivative with respect to time) and correspondent finite – difference operators suitable for experimental data processing. Generally the operators are integral and the explicit expressions of them will be published elsewhere.

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