AN ALGORITHM OF VELOCITY FIELD EVALUATION VIA TEMPERATURE FIELD MESUREMENT ON THE BASIS OF STREAMLINE-DEFINED COORDINATES

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Abstract: A new algorithm is presented for evaluating the velocity field of the heat convection flow from a vertical solid flat plate using the temperature field. Using the coordinate transformation of Cartesian coordinates, to coordinates defined via streamlines, and visible in such coordinate approximations, it is possible to express the basic flow fields in terms of the temperature gradients only. After discretization we formulate approximated sufficient finite-difference formulas to evaluate the velocity field using the experimental data.

Keywords: free convective heat transfer, isothermal surface, numerical Analysis, natural Convection, streamline Coordinate System

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1. Introduction

Analytic and experimental as well as numerical studies of free convective heat transfer from isothermal plates have been done extensively by numerous researchers due to industrial and engineering applications such as insulation electronic equipment, and nuclear reactors. There is a renewed desire in the study of the natural free heat convective flow from vertical plates to determine the velocity field of the flow by different ways as it is very useful to understand the flow better and it has many uses such as determination of the convective heat and momentum losses from devices by designers and engineers. Many research studies have been done on a different type of plates and flat surfaces with inclined orientation from vertical to horizontal.

One of these studies that represent a numerical solution of natural free convection flow velocity of isothermal plates was carried out by Bodia and Osterle [1] using the boundary layer approximation when the fluid is considered as air comes down to the analytic, asymptotic and numerical solution of Navier-Stokes and Fourier-Kirchhoff equations, together with continuity equations and the equation of state for gases. Examples of such research studies are presented in the works of Churchill and Chu [2], Martynenko *et al.* [3], Shapiro [4], Hellums *et al.* [5], Vynnycky *et al.* [6], Leble and Lewandowski [7, 8] and others. We continue the investigation of determination of the velocity-field of the flow in the vicinity of a solid plate having the temperature field that is obtained from an experiment (*cf.*, *e.g.* Lewandowski *et al.* [9]).

The mechanism of natural convection corresponds to a heat transfer associated with the movement of matter in fluids. It mainly depends on the joint action of temperature and velocity fields. The temperature changes inside the fluid or between the fluid and heated or cooled solid walls are caused by heat and momentum transfer and variations in the fluid density, which in effect begins moving under the influence of the buoyancy forces. On the other hand, fluid movement intensifies further with the energy exchange. This mutual feedback continues until it reaches specific stabilized temperature [9] in which the thermodynamic equilibrium of the temperature and velocity fields are determined. It may also result from the mass transfer when the fluid is a multi-component mixture [10].

In the first and second sections as well as in the first subsection of the third one we reproduce the results from the article [11] for the reader's convenience. The second subsection is devoted to discretization of the fields that is necessary for adequate representation of the results of measurements and, as a main task of the paper, to prepare a numerical algorithm to express the velocity field variables. A way to realize the evaluation is paved in the third subsection of the third section. It is written as a system of nonlinear difference equations to be solved by some conventional algebraic methods.

2. Mathematical formulation

The first step for any analysis is to determine the physical conditions in which the system is and formulate the phenomena description mathematically. Let us consider a flow which is generated by a heat convection transfer from an isothermal solid plate with a length L. The choice of the position of the plate comes from the typical position of the experiment which gives cross-sectional measurements of the temperature field in the xy plane for different measurements with a change of the position of the thermal IR camera in z coordinate. The plate is heated to a temperature T_w, the temperature of a non-disturbed area is T_∞. It is a surface of wide span in the z direction, the Cartesian coordinates are shown in Figure 1, marked as x and y. For example, a vertical plate lies at

$y=0$, $x\in(0,L)$. We conventionally consider a two-dimensional stationary flow of an incompressible fluid in the gravity field with acceleration g, fluid density ρ and the fluid density in the non-disturbed area is ρ_∞. Consider that the generated velocity of the fluid flow is $\overrightarrow{W}=(W_x,W_y)$ which is inclined by the angle θ to the x-axis. As for pressure p, we mark that one of the non-disturbed areas is p_∞.

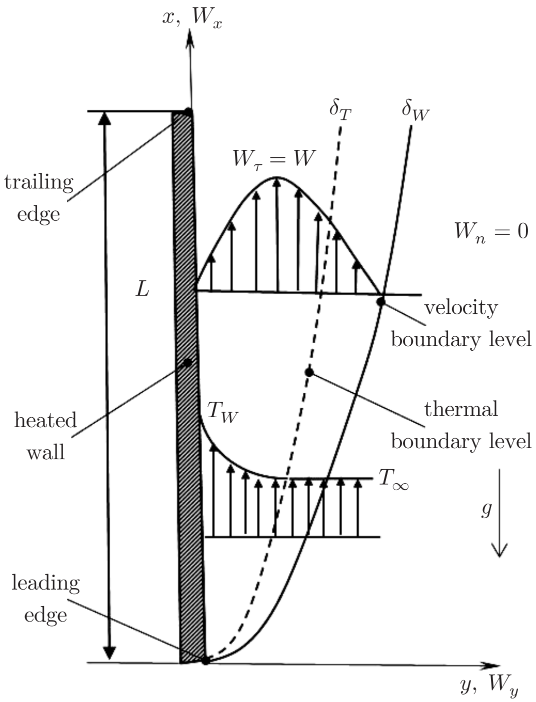

Figure 1. Schematic of flow model from heated plate

The variations of density ρ are considered as linear against temperature T. Using a reference state defined by ρ_∞, p_∞, T_∞ (these variables are linked by the fluid state equation), density variations are given by:

$$\rho=-\beta(T-T_\infty) \tag{1}$$

where $\beta=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}\right)_p=\frac{1}{T_\infty}$ is the coefficient of the thermal expansion of the fluid at constant pressure. The flow problems are governed by three conservation laws: the conservation of mass, the conservation of momentum, and the conservation of energy. These laws are the main assumptions in dealing with our present problem. The Mass conservation equation or the continuity equation present the balance of masses entering, leaving a unit volume and the change in density; for the density ρ and the velocity $\overrightarrow{W}$, the continuity equation has the form:

$$\partial_t\rho+\nabla\cdot\left(\rho\overrightarrow{W}\right)=0 \tag{2}$$

The derivatives are denoted by the symbol ∂ and by the scalar product of two vectors $\vec{a}\cdot\vec{b}$. In the steady state, the density variations are ignored in the mass continuity equation in the conditions of natural convection, liquid or gases in free convection are considered virtually incompressible; in other words, the density is not affected by the force field. In this case the continuity equation keeps the usual expression:

$$\nabla\cdot\overrightarrow{W}=0 \tag{3}$$

The conservation of momentum, or the equation of motion, is derived from the well-known Navier-Stokes equations which describe the motion of a viscous fluid through the consideration of inter-molecular forces based on the assumption that the normal and shearing stresses in a fluid are proportional to the deformation velocities. For a steady, incompressible, two-dimensional flow and assuming that viscosity is constant, the Navier-Stokes system of equations in the Cartesian coordinates has the form:

$$\overrightarrow{W}\cdot\nabla W_y=-g\beta(T-T_\infty)-\frac{1}{\rho}\nabla_y p+\nu\Delta W_y \tag{4}$$

$$\overrightarrow{W}\cdot\nabla W_x=-\frac{1}{\rho}\nabla_x p+\nu\Delta W_x \tag{5}$$

where ν is the kinematic coefficient of viscosity. These two equations express the balance of inertia forces, body forces, pressure forces and viscous forces in the x and y directions. The pressure is divided in hydrostatic pressure that is equal to the mass force $-\rho g$ and arises from the buoyancy force $-g\beta(T-T_\infty)$ due to the density variations and the dependence of the density on the temperature changes. The conservation of energy or the energy equation (Fourier-Kirchhoff equation) describes the temperature dynamics and for the incompressible fluid flow with temperature T from the vertical plate, assuming that the viscous dissipation is negligible, the energy equation takes the form:

$$\overrightarrow{W}\cdot\nabla T=a\Delta T \tag{6}$$

where a is the temperature diffusivity.

The boundary conditions follow [12]. At the heated wall ($y=0$), the no-slip and impermeability conditions, yield the following boundary condition for the momentum equation $\overrightarrow{W}=0$, $\theta=0$ and $T=T_w$. Since the quiescent fluid far away from the heated wall is not disturbed by the existence of the heated plate, the velocity at the locations away from the flat plate at ($y\to\infty$) should be zero, $\overrightarrow{W}=0,\theta=0$ and $T=T_\infty$.

Let the dimensionless variables be $\left(x^*=\frac{x}{L},\ y^*=\frac{y}{L},\ T^*=\frac{T-T_\infty}{T_w-T_\infty},\ p^*=\frac{p}{p_\infty},\ \overrightarrow{W}^*=\frac{\overrightarrow{W}^*}{W_0}\right)$ and $\left(\nu^*=\frac{\nu}{LW_o},\ a^*=\frac{a}{LW_o}\right)$, where W_0 is a reference velocity or the characteristic velocity that is unknown at this point.

After dropping the stars from the mass continuity equation we can introduce the stream function ψ as:

$$W_x=-\partial_y\psi,\qquad W_y=\partial_x\psi \tag{7}$$

Subsequently, we can rewrite the governing equations in the form based on the streamline function [7], skipping the continuity equation, that holds identically.

The solution of the two-dimensional governing equations of the flow by vertical plate model presents many difficulties in the computational formulation of the problem. One of the major factors which affects in reducing the problem formulation complexity is the choice of an appropriate coordinate system to be used as a basis for the differences in the governing equations. In the selection of such a coordinate system; one can use many options available in such selection, one of these options is simply to use the curvilinear streamline coordinate system. This selection is very appropriate and can present the governing equations in their simplest form. There are many advantages to use this coordinate system in writing the flow equations. The streamline coordinate is an orthogonal moving frame aligned locally with the streamlines while the independent variables are simply the distance along the streamlines and the distorted distance along the flow field. A direct use of the streamline coordinate for many purposes has also received attention and has been widely investigated due to its precise results. A description of some examples may be found in [13] and [14]. In addition, the main motivation by introducing the fundamental governing equations in the flow in essential form is that we may perceive simplifications and parameterizations that are not just for a particular flow but which are universal.

Let the flow be described with the coordinates (τ,n), where τ is along the streamline and n is a coordinate along the outward drawn normal to the stream line, defined by the unit vectors $\hat{\tau}=(\cos\theta,\sin\theta)$, and $\hat{n}=(-\sin\theta,\cos\theta)$. The streamline function ψ are the vector lines of the velocity field which can be defined by $\psi(x,y)=n$. The stream function also has the attribute, as

$$\psi-\psi_0=\int(W_x dy-W_y dx) \tag{8}$$

The velocity $\overrightarrow{W}$ of the fluid particle in this coordinate system is the tangent to the stream line curve, and it makes an angle θ with the x-axis. The velocity component in the normal direction to the streamline is zero $W_n=0$. Hence, we can mark the tangent component of the velocity as its module $W_\tau=W$.

Let a function $\varphi(x,y)$ which is constant across trajectories of streamlines in the two-dimensional flow be derived, so that the coordinate $\varphi(x,y)=\tau$ is introduced. From the streamline function definition, we can define the family of curves tangent to the streamlines as $y=f_1(x,n)$ and the family of orthogonal curves to the streamlines as $y=f_2(x,\tau)$. It yields a new streamline curvilinear coordinate system with the variables τ, n. The variables are connected with the Cartesian coordinate system by:

$$n=\psi(x,y), \qquad \tau=\varphi(x,y) \tag{9}$$

The family of curves tangent to the streamlines are given from (7) in the total form as

$$\partial_y\psi dx-\partial_x\psi dy=0 \tag{10}$$

The equation (10) is exact, if and only if $\Delta\psi=0$, which drives the lamellar flow condition

$$\mathrm{Curl}_z\overrightarrow{W}=0 \tag{11}$$

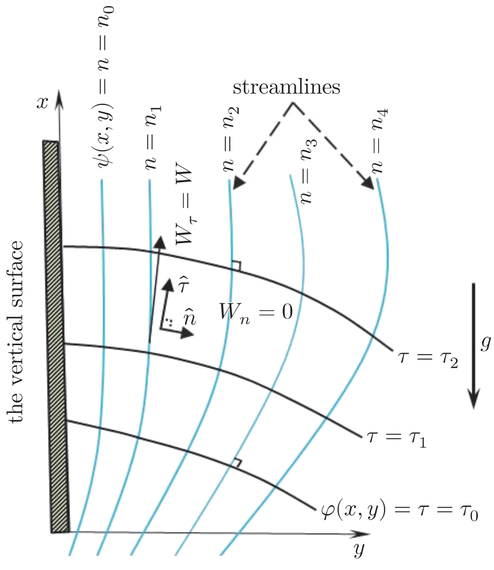

Figure 2. Streamlines of flow from a heated vertical plate

This condition leads to the velocity potential existence and concurrence with φ, which are lines; their tangents are proportional to $\nabla\psi$ so they must satisfy the ordinary differential equation $W_x dx + W_y dy = 0$. A solution to the differential equation exist in complex-lamellar flows, condition (11), but in the case of a viscous flow we consider and we are forced to generalize the description. Namely, in the two-dimensional case there exists an integration factor $\mu(x,y)$ [15], such that $(\mu W_x)dx + (\mu W_y)dy = 0 = d\varphi$ is an exact differential equation, and

$$\partial_x \varphi = \mu W_x, \qquad \partial_y \varphi = \mu W_y \tag{12}$$

A coordinate system transformation is entirely defined by its metric, a quantity which contains only the partial differential coefficients of the streamline coordinate with respect to the reference framework, in this case, the Cartesian coordinates. We will consider these coefficients completely specified if they can be written in terms of Cartesian velocity components, and it is, therefore, clear from (7) that the problem becomes one of specifying the integrating factor μ. The connection between the integration factor μ and ψ is the integrability condition:

$$\nabla(\mu\nabla\psi) = 0 \tag{13}$$

If $\mu = 1$ ($\Delta\psi = 0$), then it is equivalent to the equation of the velocity potential and this means that the term of viscosity vanishes in the Navier-Stokes equations. In our case, the variable $\mu \neq 1$ is an auxiliary variable.

The components of the metric tensor $G_{ik} = G_{ki}$ of the streamline coordinates are $G_{nn} = (\partial_n x)^2 + (\partial_n y)^2$, $G_{n\tau} = \partial_n x \partial_\tau x + \partial_n y \partial_\tau y$, and $G_{\tau\tau} = (\partial_\tau x)^2 + (\partial_\tau y)^2$ with the determinant $G = G_{nn}G_{\tau\tau} - G_{n\tau}^2$. Where $x = x(\tau, n)$ and $y = y(\tau, n)$ define the inverse transformation of (9). From the contravariant vector transformation and using Christoffel symbols of the second kind [15], the components of the metric tensor are:

$$G_{nn} = \frac{1}{W^2}, \qquad G_{\tau\tau} = \left(\frac{W_x}{\varphi_x}\right)^2 \frac{1}{W^2}, \qquad G = \left(\frac{W_x}{\varphi_x}\right)^2 \frac{1}{W^4} \tag{14}$$

The procedure adopted in deriving the governing transformed equations in the streamline coordinate system follows [11]. It is to write down the original equations in Cartesian and in dimensionless form, rewrite the equations in the general vector form, replace the partial derivatives by the partial covariant derivatives and then recover the physical components. The condition (14)gives the relation between μ and φ_x which can be written as $\mu = (\varphi_x / W\cos\theta)$, this relation can be used to re-introduce the governing equation in terms of the integration factor μ. This system, taking into account our considerations concerning the boundary, is considered as the formulation of the problem to be solved in new independent coordinates n, τ. Applying the non-singular perturbation theory [16] to the system of equations by a small parameter, in the current formulas below, after the approximation choice, the parameter is chosen conventionally equal to the unity. The transport of the momentum of the fluid particle is similar but it is determined by buoyancy and viscosity forces that act in different directions. The module of velocity is changed essentially along the perpendicular direction to streamlines while its angle of inclination changes in the opposite direction. After the first approximation and integration, the energy equation and the Navier-Stokes system of equations give:

$$\mu(\tau,n) = \frac{a\partial_n T(\tau,n)}{\int_{n_0}^{n} \partial_\tau T(\tau,n)dn} \tag{15}$$

and

$$W(\tau,n) = \exp\left(-\int_{\tau_0}^{\tau} \frac{\partial_n \theta(\tau,n)}{\mu(\tau,n)} d\tau\right) \tag{16}$$

where

$$\theta(\tau,n) = \int_{\tau_0}^{\tau} \left(\frac{1}{a} - \frac{\partial_n^2 T(\tau,n)}{\mu(\tau,n)\partial_n T(\tau,n)}\right) d\tau \tag{17}$$

For more details about the procedures, see [11]. The functions (16) and (17) allow determining the heat convection velocity field on the basis of the temperature gradients.

3. The velocity field

3.1. Differential-geometric approach to the flow

The procedures for evaluating the free convection velocity field use the ordinary differential equations which describe the geometry of the flow corresponding with the streamline coordinate system. The differential equations link between the temperature gradients in both coordinate systems. The differentials in both coordinate systems are connected by elements of differential geometry. Let $d\tau$ be an elementary distance along the streamline and dn be an elementary distance orthogonal to the streamline. From the equation (9), we have $\partial_x n = \partial_x \psi$ and $\partial_y n = \partial_y \psi$, hence:

$$dn = \partial_y \psi dx + \partial_x \psi dy \tag{18}$$

Using the relation between the partial derivatives of the stream function and the velocity components (7) $\partial_y \psi = -W_x$ and $\partial_x \psi = W_y$ with the definitions of $W_x = W\cos\theta$ and $W_y = W\sin\theta$, the last equation takes the form:

$$dn = W\left(\sin\theta dx - \cos\theta dy\right) \tag{19}$$

Similarly, for the distance along the streamline $d\tau$, from (9), we have $\partial_x \tau = \varphi_x$ and $\partial_y \tau = \varphi_y$, we can write:

$$d\tau = \varphi_x dx + \varphi_y dy \tag{20}$$

By the definition of the generalized velocity potential via the orthogonality condition using the differential equations (12) of the integration factor we have $\varphi_x = \mu W_x$ and $\varphi_y = \mu W_y$, then the equation for $d\tau$ takes the form:

$$d\tau = \mu W\left(\sin\theta dx + \cos\theta dy\right) \tag{21}$$

Now, we have two equations (19) and (21) which describe the geometry in the streamline coordinate system, where μ, θ, and W can be calculated by the temperature field from the functions (15), (17), and (16). The temperature is expressed as a function $T(\tau,n)$ in the streamline coordinate system, we can write

$$dT(\tau,n) = \partial_n T dn + \partial_\tau T d\tau \tag{22}$$

From the equations (19) and (21), we have:

$$dT = W\left(\partial_\tau T \mu\cos\theta + \partial_n T\sin\theta\right)dx + W\left(\partial_\tau T\mu\sin\theta - \partial_n T\cos\theta\right)dy \tag{23}$$

Using the fact that the partial derivatives with respect to the Cartesian coordinate system are related to partial derivatives with respect to the streamline coordinate system $dT(\tau,n) = dT(x,y)$, by the chain rule, we have the relations which link between the temperature gradients in both the coordinate systems, as:

$$\partial_n T = \frac{\sin(\theta)\partial_x T - \cos(\theta)\partial_y T}{W} \tag{24}$$

and

$$\partial_\tau T = \frac{\cos(\theta)\partial_x T + \sin(\theta)\partial_y T}{\mu W} \tag{25}$$

The same relations can be obtained if we use the direct link of the partial derivatives of the two coordinate systems using Christoffel symbols (14) of the second kind.

3.2. Discrete form of the velocity field

Let the xy plane be covered by a mesh the nodes of which (i,j), $0 \le i \le I$, $0 \le j \le J$ corresponding to the points $x = x_i$, $y = y_j$. In these notations, the

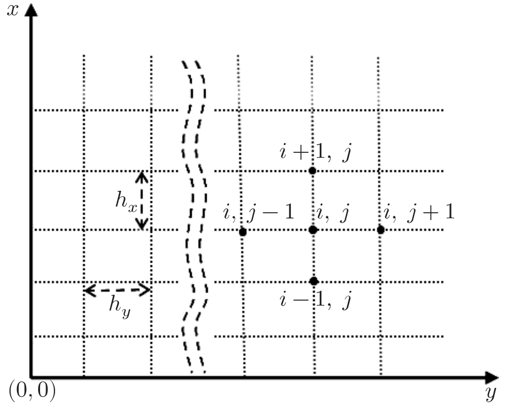

Figure 3. Computational grid portion showing numbering systems in xy Cartesian coordinate system

boundaries are at $x=x_0$, $x=x_I$, $y=y_0$ and $y=y_J$. Variable node spacings are introduced by:

$$h_x = x_{i+1} - x_i, \qquad h_y = y_{j+1} - y_j \qquad (0 \le i \le I, 0 \le j \le J) \tag{26}$$

here $h_x = 1/I$ and $h_y = 1/J$. Let the temperature field at an arbitrary point (x_i, y_j) be $T(x_i, y_j) = \hat{T}^{i,j}$. The temperature gradients can be calculated in the Cartesian coordinate system at the point (x_i, y_j) from the backward difference equation. The point (x_i, y_j) in the Cartesian coordinate system is shifted in the streamline coordinate system to the point (τ_i, n_j) as we are evaluating the velocity field in different coordinate systems but we are still calculating the velocity field in the same space. Let the streamline along distance $d\tau$ and the orthogonal distance dn be given from shifting the point (τ_i, n_j) by $s_\tau^{i,j}$, $s_n^{i,j}$ in the order which is shown in the numbering system of the curvilinear streamline coordinate system, see Figure 4, where

$$s_\tau^{i+1,j} = \tau_{i+1} - \tau_i, \qquad s_n^{i,j+1} = n_{j+1} - n_j \qquad (0 \le i \le I, 0 \le j \le J) \tag{27}$$

Thus, we can define an arbitrary point (τ_i, n_j) on the streamline coordinate system using the node spacing variables $s_\tau^{i,j}$, $s_n^{i,j}$ by:

$$(\tau_i, n_j) = \left(\sum_{l=1}^{i} s_\tau^{l,j}, \sum_{f=1}^{j} s_n^{i,f} \right) \tag{28}$$

Restricting our consideration by a vertical plate, using Taylor expansions with the first order approximation, we have $\sin(\theta) \approx \theta, \cos(\theta) \approx 1$, and the velocity field function (16), in the first order approximation, takes the form:

$$W(n,\tau) \approx 1 - \int_{\tau_0}^{\tau} \frac{\partial_n \theta(\tau, n)}{\mu(\tau, n)} d\tau \tag{29}$$

From the previous approximations the geometry flow relations (19) and (21) can be written as:

$$dn \approx W\Big(\theta dx - dy\Big) \tag{30}$$

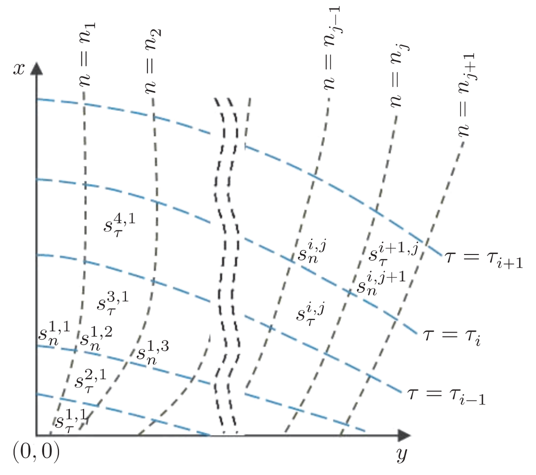

Figure 4. Computational grid portion showing streamline coordinate system numbering systems

and

$$d\tau \approx \mu W\Big(dx+\theta dy\Big) \tag{31}$$

If we consider $dx \approx \Delta x = h_x$ and $dy \approx \Delta y = h_y$, then the geometry flow equations in the discrete form are:

$$s_n^{i,j} \approx W^{i,j-1}\Big(\theta^{i,j-1}h_x - h_y\Big) \tag{32}$$

and

$$s_\tau^{i,j} \approx \mu^{i-1,j}W^{i-1,j}\Big(h_x + \theta^{i-1,j}h_y\Big) \tag{33}$$

Then

$$(\tau_i, n_j) = \left(\sum_{l=1}^{i}\mu^{l-1,j}W^{l-1,j}\Big(h_x+\theta^{l-1,j}h_y\Big), \sum_{f=1}^{j}W^{i,f-1}\Big(\theta^{i,f-1}h_x-h_y\Big)\right) \tag{34}$$

For every i and j such that $(0 < i < I,\ 0 < j < J)$. Consequently, we can approximate (24) and (25) as:

$$\partial_n T \approx \frac{\theta\partial_x T - \partial_y T}{W} \tag{35}$$

$$\partial_\tau T \approx \frac{\partial_x T + \theta\partial_y T}{\mu W} \tag{36}$$

In the discrete form, at an arbitrary point (τ_i, n_j), we can define the temperature gradient n-component as $\zeta^{i,j} = (\partial_n T)_{\tau=\tau_i, n=n_j}$, hence,

$$\zeta^{i,j} \approx \frac{\theta^{i,j}(\partial_x T)^{i,j} - (\partial_y T)^{i,j}}{W^{i,j}} \tag{37}$$

and we can define the temperature gradient component with respect to τ at an arbitrary point (τ_i, n_j) as $\eta^{i,j} = (\partial_\tau T)_{\tau=\tau_i, n=n_j}$, then,

$$\eta^{i,j} \approx \frac{(\partial_x T)^{i,j} + \theta^{i,j}(\partial_y T)^{i,j}}{\mu^{i,j}W^{i,j}} \tag{38}$$

We define the integrating factor at an arbitrary point (τ_i, n_j), from (15) by:

$$\mu^{i,j} = \frac{a\partial_n T(\tau,n)_{\tau=\tau_i,n=n_j}}{\int_{n_0}^{n_j} \partial_\tau T(\tau,n)_{\tau=\tau_i} dn} \tag{39}$$

By using the left Riemann sum, we have, approximately:

$$\mu^{i,j} \approx \frac{a\zeta^{i,j}}{\sum\limits_{f=1}^{j} \eta^{i,f} s_n^{i,f}} \tag{40}$$

For every i and j such that $(0 < i < I,\ 0 < j < J)$. Similarly, for (17) at an arbitrary point (τ_i, n_j), we have:

$$\theta^{i,j} = \int\limits_{\tau_0}^{\tau_i} \left(-\frac{\partial_n^2 T(\tau,n)}{\mu(\tau,n)\partial_n T(\tau,n)} + \frac{1}{a} \right)_{n=n_j} d\tau \tag{41}$$

We can write the second derivatives of the temperature field respective to n by the backward difference equation using (37) and using (40), and using the Riemann sum, we have:

$$\theta^{i,j} \approx \sum_{l=1}^{i} \left(\frac{(\zeta^{l,j-1}/\zeta^{l,j}) - 1}{\mu^{l,j} s_n^{l,j}} + \frac{1}{a} \right) s_\tau^{l,j} \tag{42}$$

For every i and j such that $(0 < i < I,\ 0 < j < J)$ the velocity field function (29) at (τ_i, n_j) can be written as:

$$W^{i,j} \approx 1 - \int\limits_{\tau_0}^{\tau_i} \left(\frac{\partial_n \theta(\tau,n)}{\mu(\tau,n)} \right)_{n=n_j} d\tau \tag{43}$$

Hence,

$$\begin{aligned} W^{i,j} \approx 1 - \sum_{l=1}^{i} \frac{s_\tau^{l,j}}{s_n^{l,j}\mu^{l,j}} \sum_{t=1}^{l} \Bigg[& \left(\frac{(\zeta^{t,j-1}/\zeta^{t,j}) - 1}{\mu^{t,j} s_n^{t,j}} + \frac{1}{a} \right) s_\tau^{t,j} - \\ & \left(\frac{(\zeta^{t,j-2}/\zeta^{t,j-1}) - 1}{\mu^{t,j-1} s_n^{t,j-1}} + \frac{1}{a} \right) s_\tau^{t,j-1} \Bigg] \end{aligned} \tag{44}$$

for every i and j such that $(0 < i < I,\ 0 < j < J)$.

3.3. System of equations for temperature gradient in streamline coordinates

The procedures for evaluating the heat convection velocity field depend on the gradients of the temperature with respect to the streamline coordinates, these gradients can be calculated from (37) and (38). These two equations contain the velocity function. If we substitute the numerical function for evaluating the velocity field, as well, the integrability constant into the temperature gradients equations, we have nonlinear equations in the gradients only as:

$$\zeta^{i,j}\left[1-\sum_{l=1}^{i}\frac{s_\tau^{l,j}}{s_n^{l,j}\mu^{l,j}}\sum_{f=1}^{l}\left[\left(\frac{(\zeta^{f,j-1}/\zeta^{f,j})-1}{\mu^{f,j}s_n^{f,j}}+\frac{1}{a}\right)s_\tau^{f,j}-\left(\frac{(\zeta^{f,j-2}/\zeta^{f,j-1})-1}{\mu^{f,j-1}s_n^{f,j-1}}+\frac{1}{a}\right)s_\tau^{f,j-1}\right]\right]=(\partial_xT)^{i,j}\sum_{l=1}^{i}\left(\frac{(\zeta^{l,j-1}/\zeta^{l,j})-1}{\mu^{l,j}s_n^{l,j}}+\frac{1}{a}\right)s_\tau^{l,j}-(\partial_yT)^{i,j} \tag{45}$$

and

$$\eta^{i,j}\left[1-\sum_{l=1}^{i}\frac{s_\tau^{l,j}}{s_n^{l,j}\mu^{l,j}}\sum_{f=1}^{l}\left[\left(\frac{(\zeta^{f,j-1}/\zeta^{f,j})-1}{\mu^{f,j}s_n^{f,j}}+\frac{1}{a}\right)s_\tau^{f,j}-\left(\frac{(\zeta^{f,j-2}/\zeta^{f,j-1})-1}{\mu^{f,j-1}s_n^{f,j-1}}+\frac{1}{a}\right)s_\tau^{f,j-1}\right]\right]\mu^{i,j}=(\partial_xT)^{i,j}+(\partial_yT)^{i,j}\sum_{l=1}^{i}\left(\frac{(\zeta^{l,j-1}/\zeta^{l,j})-1}{\mu^{l,j}s_n^{l,j}}+\frac{1}{a}\right)s_\tau^{l,j} \tag{46}$$

These equations allow us to express the temperature gradients $\zeta^{i,j}$ and $\eta^{i,j}$ as functions that solve the system (45), (46):

$$\zeta^{i,j}=\Phi_1\Big((\partial_xT)^{i,j},\,(\partial_yT)^{i,j}\Big) \tag{47}$$

and

$$\eta^{i,j}=\Phi_2\Big((\partial_xT)^{i,j},\,(\partial_yT)^{i,j}\Big) \tag{48}$$

The functions $s_\tau^{i+1,j}$, $s_n^{i,j+1}$, that define the grid in streamline coordinates are defined by (32) and (33). The expressions contain the velocity module, the angle and the integrating factor, hence the algorithm should be built so as to run the points by layers, taking the values from a previous layer. A starting point may be chosen as the origin of the coordinate systems.

The steps of building the algorithm start from the temperature measurements table for the vicinity of the heated plate including the boundary conditions, these measurements construct a discrete form of the temperature field and based on these temperatures we introduce the spacing distances h_x and h_y between the points in the numbering systems of Cartesian coordinates, then we calculate using the difference equations for the temperature gradients in the same coordinate system by the direct relations between the temperature gradients $\zeta^{i,j}$, $\eta^{i,j}$ and the gradients in the Cartesian coordinates (37) and (38). The integrability constant at an arbitrary point (τ_i,n_j) is a function in the temperature gradients $\zeta^{i,j}$, $\eta^{i,j}$ which can be calculated directly from them. Similarly, after calculating the integrability constant, the velocity angles can be calculated from (42) at the same point. Consequently, the velocity field can be evaluated directly from (44) based on the temperature field, this completes the full description.

4. Conclusion

The numerical functions and equations, which are obtained in the last section, form the full necessary and sufficient mathematical formulas to evaluate the heat convection velocity field based on the temperature in the vicinity of the vertical plate. The resulting scheme, in a perspective, may be extended to solve the discrete version of a full basic Fourier-Kirchhoff-Navies-Stokes system with given boundary conditions. It is also important to prove theorems about stability and convergence of the proposed discrete algorithm, when $h_{x,y} \to 0$.

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