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MATRIX GREEN'S FUNCTION OF DOUBLE-DIFFUSIVITY PROBLEM AND ITS APPLICATIONS TO PROBLEMS WITH INNER POINT SOURCE

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Abstract: The matrix Green's function of the initial-boundary value problem of admixture double-diffusivity is defined. The initial-boundary value problem with a point source is formulated for the matrix elements for determination of the matrix Green's function. Formulae for matrix elements are obtained and the behavior of Green's functions is investigated. It is shown that the surface generated by the Green's function has a typical sharp peak in the vicinity of the point of action of the point mass source, and in the vicinity of the top boundary of the layer, the values of the second element of the Green's function are times higher than the values of the first one the state of which is corresponding to the quick migration way. On this basis the solutions of the initial-boundary value problems under the action of the internal point source of mass are found. The cases of the deterministic source as well as stochastic ones under uniform and triangular distributions of the coordinate of the mass source location are considered.

Keywords: Green's function, double-diffusivity, initial-boundary value problem, point mass source, random coordinate

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1. Introduction

One of the classic methods of solving the initial-boundary value problems of mathematical physics, which is widely used in practice, is the method of Green's functions, see [1–4]. Obtaining such a solution of initial-boundary value problems is based on the third Green's formula, see [5, 6]. In turn, Green's functions are solutions of the corresponding problems with a point source under the zero initial and boundary conditions, see [7–10]. Then Green's functions have an autonomous significance, but they are often used for constructing solutions of nonhomogeneous problems of mathematical physics. Green's functions are of interest for finding solutions of transfer problems in bodies with microstructures, especially under the occurrence of internal sources. Moreover, in particle physics, Green's functions are used as propagators in Feynman diagrams for the description of wave processes, see [11, 12]. Also, Green's functions are widely used in the application of the scattering theory in the physics of solids (X-ray, calculations of electronic spectra of metallic materials, *etc.*), see [4, 2].

The paper touches on the definition, construction and investigation of the Green's function of the problem of double-diffusivity in a layer with two ways for migration. On this basis the concentrations of migrating particles under the action of both internal deterministic and random point sources of mass are obtained.

2. Matrix Green's function of double-diffusivity initial-boundary value problem

Let the vector-function

$$\mathbf{c}(\xi,\tau)=\begin{pmatrix} c_1(\xi,\tau) \\ c_2(\xi,\tau) \end{pmatrix} \tag{1}$$

be such that its elements $c_i(\xi,\tau)$, $i=1,2$, are continuous in the variables ξ and τ in the domain $R=\{(\xi,\tau):\xi\in[0;\xi_0],\ \tau\in\mathfrak{R}_+\}$, satisfy the Lipschitz property in the variable ξ in the domain R with constant l, see [13, 14].

Consider the linear initial-boundary value problem for the system of coupled partial differential equations of the 2^{nd} order

$$\mathbf{L}[\mathbf{c}(\xi,\tau)]=\mathbf{F}(\xi,\tau) \tag{2}$$

where

$$\mathbf{L}=\begin{pmatrix} L_1^{c_1}(\xi,\tau) & L_1^{c_2}(\xi,\tau) \\ L_2^{c_1}(\xi,\tau) & L_2^{c_2}(\xi,\tau) \end{pmatrix} \tag{3}$$

is the matrix partial differential operator, in which

$$\begin{aligned} &L_1^{c_1}(\xi,\tau)=\frac{\partial}{\partial\tau}-\frac{\partial^2}{\partial\xi^2}+\tilde{a}_{11} && L_1^{c_2}(\xi,\tau)\equiv L_1^{c_2}(\xi)=-d_1\frac{\partial^2}{\partial\xi^2}-\tilde{a}_{12} \\ &L_2^{c_1}(\xi,\tau)\equiv L_2^{c_1}(\xi)=-d_2\frac{\partial^2}{\partial\xi^2}-\tilde{a}_{21} && L_2^{c_2}(\xi,\tau)=\frac{\partial}{\partial\tau}-d\frac{\partial^2}{\partial\xi^2}+\tilde{a}_{22} \end{aligned} \tag{4}$$

(the upper indexes of elements of the matrix operator indicate on what function the given element of the matrix operator works);

$$\mathbf{F}(\xi,\tau)=\begin{pmatrix}F_1(\xi,\tau)\\F_2(\xi,\tau)\end{pmatrix} \tag{5}$$

is the vector-function of sources, where $F_i(\xi,\tau)\in L_2 \vee F_i\in D(\mathfrak{R}^2)\vee F_i\in\langle\Omega,\sigma,P\rangle$.

Let the initial and boundary conditions for a layer of thickness ξ_0 be imposed

$$\mathbf{c}(\xi,\tau)|_{\tau=0}=\begin{pmatrix}0\\0\end{pmatrix};\quad \mathbf{c}(\xi,\tau)|_{\xi=0}=\mathbf{c}_0=\begin{pmatrix}c_0^{(1)}\\c_0^{(2)}\end{pmatrix};\quad \mathbf{c}(\xi,\tau)|_{\xi=\xi_0}=\begin{pmatrix}0\\0\end{pmatrix} \tag{6}$$

where $c_0^{(i)}\in L_2$.

If $|\mathbf{c}(\xi,\tau)-\mathbf{c}_0(\xi,\tau)|\leq\beta$ for $\forall t\in\mathfrak{R}_+$, $\forall\xi\in[0;\xi_0]$ (β is the known positive constant) then there exists a single classical solution of the problem (2) and (6), see [6].

Definition A: The Green's function of the problem (2) and (6) is called the matrix function

$$\mathbf{G}(\xi,\xi';\tau,\tau')=\begin{pmatrix}G_1(\xi,\xi';\tau,\tau') & 0\\ 0 & G_2(\xi,\xi';\tau,\tau')\end{pmatrix} \tag{7}$$

defining in the four-dimensional domain $K=K_1\times K_2=\{(\xi,\xi';\tau,\tau')|\ \xi,\xi'\in[0;\xi_0];\ \tau,\tau'\in\mathfrak{R}_+:\tau'<\tau\}$ and satisfies such conditions:

1. in K the matrix function $\mathbf{G}(\xi,\xi';\tau,\tau')$ is continuous and has continuous derivatives with time τ;
2. for arbitrary $\xi'\in[0;\xi_0]$, $\tau'\in\mathfrak{R}_+$ $\mathbf{G}(\xi,\xi';\tau,\tau')$ has continuous derivatives of the first and second orders with respect to the variable ξ in each of the intervals $[0;\xi')$ and $(\xi';\xi_0]$, and derivative of the first order at the point $\xi=\xi'$ has a jump that is equal to one:
$$\frac{\partial}{\partial\xi}\mathbf{G}(\xi+0,\xi';\tau,\tau')-\frac{\partial}{\partial\xi}\mathbf{G}(\xi-0,\xi';\tau,\tau')=1 \tag{8}$$
3. $\mathbf{G}(\xi,\xi';\tau,\tau')=0$ for $\tau\leq\tau'$;
4. in each of the intervals $[0;\xi')$ and $(\xi';\xi_0]$ for $\tau'\leq\tau$ the function $\mathbf{G}(\xi,\xi';\tau,\tau')$ as a function of the variable ξ is a solution of the homogeneous equation
$$\mathbf{L}[\mathbf{G}(\xi,\xi';\tau,\tau')]=0 \tag{9}$$
5. $\mathbf{G}(\xi,\xi';\tau,\tau')$ as a function of the variables ξ and τ satisfies zero initial and boundary conditions of type (6).

Note that in the Banach space $\bar{C}^{(n,q)}(D^p)$ of the vector-functions $\mathbf{c}(\xi,\tau)$ with norm

$$\|\mathbf{c}\|_{\bar{C}^{(\bar{n},q)}(D^p)}=\sum_{j=1}^{2}\|c_j\|_{\bar{C}^{(n_j,2)}(D^p)} \tag{10}$$

where $D^p\equiv K'=\{(\xi,\xi';\tau,\tau')|\ \xi,\xi'\in[0;\xi_0];\tau,\tau'\in[0;T]:\tau'\leq\tau\}$, $\bar{n}=(n_1,n_2)$ and $n_1=n_2=1$, for the uniqueness of the solution of the initial-boundary value problem (2) and (6) it is necessary and sufficient that the fundamental system of equations does not have solutions in integers, see [14].

Notice that it is possible to give an identical definition of the matrix Green's function in terms of a solution of the problem with a point source, *i.e.* the Green's function is an integral kernel that can be used to solve a system of partial differential equations, see [15–17].

Definition B: Let the matrix partial differential operator **L** (3) acting on the collection of distributions over a subset Ω of some Euclidean space $\mathfrak{R}^n$ be given. A Green's function (7) at the point $(\xi',\tau')\in\Omega$ corresponding to **L** is any solution of

$$\mathbf{LG}(\xi,\xi';\tau,\tau')=\hat{\delta}(\xi-\xi';\tau-\tau') \tag{11}$$

where

$$\hat{\delta}(\xi-\xi';\tau-\tau')=\begin{pmatrix}\delta(\xi-\xi')\delta(\tau-\tau')\\ \delta(\xi-\xi')\delta(\tau-\tau')\end{pmatrix} \tag{12}$$

denotes the vector delta function.

To find the matrix Green's function $\mathbf{G}(\xi,\xi';\tau,\tau')$, we formulate the initial-boundary value problem with a point source for the matrix elements, namely

$$\begin{gathered}\frac{\partial G_1(\xi,\xi';\tau,\tau')}{\partial\tau}-D_0\frac{\partial^2 G_1(\xi,\xi';\tau,\tau')}{\partial\xi^2}-D_1\frac{\partial^2 G_2(\xi,\xi';\tau,\tau')}{\partial\xi^2}+\\ \tilde{a}_{11}G_1(\xi,\xi';\tau,\tau')-\tilde{a}_{12}G_2(\xi,\xi';\tau,\tau')=\delta(\xi-\xi')\delta(\tau-\tau')\\ \frac{\partial G_2(\xi,\xi';\tau,\tau')}{\partial\tau}-D_2\frac{\partial^2 G_1(\xi,\xi';\tau,\tau')}{\partial\xi^2}-D\frac{\partial^2 G_2(\xi,\xi';\tau,\tau')}{\partial\xi^2}-\\ \tilde{a}_{21}G_1(\xi,\xi';\tau,\tau')+\tilde{a}_{22}G_2(\xi,\xi';\tau,\tau')=\delta(\xi-\xi')\delta(\tau-\tau')\end{gathered} \tag{13}$$

under zero initial and boundary conditions

$$G_1(\xi,\xi';\tau,\tau')|_{\tau=0}=G_2(\xi,\xi';\tau,\tau')|_{\tau=0}=0 \tag{14}$$

$$G_1(\xi,\xi';\tau,\tau')|_{\xi=0}=G_2(\xi,\xi';\tau,\tau')|_{\xi=0}=0 \tag{15}$$

$$G_1(\xi,\xi';\tau,\tau')|_{\xi=\xi_0}=G_2(\xi,\xi';\tau,\tau')|_{\xi=\xi_0}=0 \tag{16}$$

To solve the problem (13)–(16) we apply Laplace's integral transformation with respect to time to this problem ($\tau\to s$, $G_j(\xi,\xi';\tau,\tau')\to\bar{g}_j(\xi,\xi';s,\tau')$). Then in images we obtain the following system of ordinary differential equations

$$\begin{gathered}(s+\tilde{a}_{11})\bar{g}_1(\xi,\xi';s,\tau')-\tilde{a}_{12}\bar{g}_2(\xi,\xi';s,\tau')-D_0\frac{\partial^2\bar{g}_1(\xi,\xi';s,\tau')}{\partial\xi^2}-\\ D_1\frac{\partial^2\bar{g}_2(\xi,\xi';s,\tau')}{\partial\xi^2}=\delta(\xi-\xi')e^{-s\tau}\\ (s+\tilde{a}_{22})\bar{g}_2(\xi,\xi';s,\tau')-\tilde{a}_{21}\bar{g}_1(\xi,\xi';s,\tau')-D_2\frac{\partial^2\bar{g}_1(\xi,\xi';s,\tau')}{\partial\xi^2}-\\ D\frac{\partial^2\bar{g}_2(\xi,\xi';s,\tau')}{\partial\xi^2}=\delta(\xi-\xi')e^{-s\tau}\end{gathered} \tag{17}$$

under zero boundary conditions

$$\begin{aligned} \bar{g}_1(\xi,\xi';s,\tau')|_{\xi=0} = \bar{g}_2(\xi,\xi';s,\tau')|_{\xi=0} = 0 \\ \bar{g}_1(\xi,\xi';s,\tau')|_{\xi=\xi_0} = \bar{g}_2(\xi,\xi';s,\tau')|_{\xi=\xi_0} = 0 \end{aligned} \tag{18}$$

We apply Fourier's sin-transformation, see [18], to the obtained boundary value problem ($\xi \to y_n = n\pi/\xi_0$, $\bar{g}_j(\xi,\xi';s,\tau') \to g_j(y_n,\xi';s,\tau')$). Here we take into account the sin-transformation from the Dirac δ-function

$$\int_0^{\xi_0} \delta(\xi-\xi')\sin(y_n\xi)d\xi = \sin(y_n\xi') \tag{19}$$

Then, the problem (17) and (18) is reduced to the system of algebraic equations

$$\begin{aligned} \left[s+\tilde{a}_{11}+D_0y_n^2\right] g_1(y_n,\xi';s,\tau') + \left[D_1y_n^2-\tilde{a}_{12}\right] g_2(y_n,\xi';s,\tau') = \sin(y_n\xi')e^{-s\tau'} \\ \left[d_2y_n^2-\tilde{a}_{21}\right] g_1(y_n,\xi';s,\tau') + \left[s+Dy_n^2+\tilde{a}_{22}\right] g_2(y_n,\xi';s,\tau') = \sin(y_n\xi')e^{-s\tau'} \end{aligned} \tag{20}$$

For obtaining the solution of the system of equations (20) we find its principal determinant

$$\Delta = \begin{vmatrix} s+\tilde{a}_{11}+D_0y_n^2 & D_1y_n^2-\tilde{a}_{12} \\ D_2y_n^2-\tilde{a}_{21} & s+Dy_n^2+\tilde{a}_{22} \end{vmatrix} = s^2+s\eta_1+\eta_2 = (s-s_1)(s-s_2) \tag{21}$$

where $\eta_1 = y_n^2(D+D_0)+\tilde{a}_{11}+\tilde{a}_{22}$; $\eta_2 = y_n^4(D_0D-D_1D_2)+y_n^2(\tilde{a}_{11}D+\tilde{a}_{22}D_0+\tilde{a}_{12}D_2+\tilde{a}_{21}D_1)+\tilde{a}_{11}\tilde{a}_{22}-\tilde{a}_{12}\tilde{a}_{21}$.

Then, we find the auxiliary determinants

$$\begin{aligned} \Delta_1 = \sin(y_n\xi')e^{-s\tau'} \begin{vmatrix} 1 & D_1y_n^2-\tilde{a}_{12} \\ 1 & s+\tilde{a}_{22}+Dy_n^2 \end{vmatrix} = \\ \sin(y_n\xi')e^{-s\tau'}\left[s+y_n^2(D-D_1)+\tilde{a}_{22}+\tilde{a}_{12}\right] \\ \Delta_2 = \sin(y_n\xi')e^{-s\tau'} \begin{vmatrix} s+D_0y_n^2+\tilde{a}_{11} & 1 \\ -\tilde{a}_{21}+D_2y_n^2 & 1 \end{vmatrix} = \\ = \sin(y_n\xi')e^{-s\tau'}\left[s+y_n^2(D_0-D_2)+\tilde{a}_{11}+\tilde{a}_{21}\right] \end{aligned} \tag{22}$$

Let us introduce the denotations

$$A_1 = y_n^2(D-D_1)+\tilde{a}_{22}+\tilde{a}_{12}; \qquad A_2 = y_n^2(D_0-D_2)+\tilde{a}_{11}+\tilde{a}_{21} \tag{23}$$

Then, the solution of the system of equations (20) takes the form

$$\begin{aligned} g_1(y_n,\xi';s,\tau') = \frac{\Delta_1}{\Delta} = \frac{\sin(y_n\xi')e^{-s\tau'}(s-A_1)}{(s-s_1)(s-s_2)} \\ g_2(y_n,\xi';s,\tau') = \frac{\Delta_2}{\Delta} = \frac{\sin(y_n\xi')e^{-s\tau'}(s-A_2)}{(s-s_1)(s-s_2)} \end{aligned} \tag{24}$$

Now we use the formulae of the inverse Laplace's transform, see [19]

$$\begin{aligned} L^{-1}\left[\frac{1}{(s-s_1)(s-s_2)}\right] = \frac{1}{s_1-s_2}\left[e^{s_1\tau}-e^{s_2\tau}\right] \\ L^{-1}\left[\frac{s}{(s-s_1)(s-s_2)}\right] = \frac{1}{s_1-s_2}\left[s_1e^{s_1\tau}-s_2e^{s_2\tau}\right] \end{aligned} \tag{25}$$

To this we apply the theorem of lateness, see [20]

$$L^{-1}\left[e^{-bs}f(s)\right]=F(\tau-b)\theta(\tau-b),\qquad b>0 \tag{26}$$

Then, we have

$$\begin{aligned}\bar{g}_1(\xi,\xi';s,\tau')&=\frac{\theta(\tau-\tau')\sin(y_n\xi')}{s_1-s_2}\left[(s_1+A_1)e^{s_1\tau}-(s_2+A_1)e^{s_2\tau}\right]\\ \bar{g}_2(\xi,\xi';s,\tau')&=\frac{\theta(\tau-\tau')\sin(y_n\xi')}{s_1-s_2}\left[(s_1+A_2)e^{s_1\tau}-(s_2+A_2)e^{s_2\tau}\right]\end{aligned} \tag{27}$$

Here $\theta(\tau-\tau')$ is the unit Heaviside function, see [21]

Now we apply the inverse Fourier's transform, see [18]. Then we have

$$\begin{gathered}G_1(\xi,\xi';\tau,\tau')=\frac{2}{\xi_0}\theta(\tau-\tau')\sum_{n=1}^{\infty}\frac{\sin(y_n\xi)\sin(y_n\xi')}{s_1-s_2}\times\\ \left[(s_1+A_1)e^{s_1(\tau-\tau')}-(s_2+A_1)e^{s_2(\tau-\tau')}\right]\end{gathered} \tag{28}$$

$$\begin{gathered}G_2(\xi,\xi';\tau,\tau')=\frac{2}{\xi_0}\theta(\tau-\tau')\sum_{n=1}^{\infty}\frac{\sin(y_n\xi)\sin(y_n\xi')}{s_1-s_2}\times\\ \left[(s_1+A_2)e^{s_1(\tau-\tau')}-(s_2+A_2)e^{s_2(\tau-\tau')}\right]\end{gathered} \tag{29}$$

Thus, we have obtained the elements of the matrix Green's function of the problem of double-diffusivity of decaying substance in a layer.

Note that the obtained series in in the expressions for the Green's functions (29) when $\tau\to\infty$ to be convergent, the conditions $s_1<0$ and $s_2<0$ must be fulfilled, see [22]. Taking into account the structure of expressions s_1 and s_2 and also that $\eta_1>0$, we obtain $s_1<0$. Analyzing the conditions under which s_2 takes on negative values, we obtain the constraint, see [22]

$$D_1D<D_0D_2. \tag{30}$$

A numerical analysis of the Green's functions $G_1(\xi,\xi';\tau,\tau')$ and $G_2(\xi,\xi';\tau,\tau')$ is carried out for such basic values of the problem parameters as: $D=0.1$, $D_0=1$, $D_1=D_2=0$; $\tilde{a}_{11}=4$, $\tilde{a}_{12}=1$, $\tilde{a}_{21}=2.2$, $\tilde{a}_{22}=2.6$. The characteristic surfaces that the Green's functions form, obtained by the formulae (29) are illustrated in Figures 1 and 2. The functions $G_1(\xi,\xi';\tau,\tau')$ in Figure 1 and $G_2(\xi,\xi';\tau,\tau')$ in Figure 2 are shown for the point $(\xi';\tau')=(1;0.01)$ (Figure (a)), $(\xi';\tau')=(2;0.01)$ (Figure (b)), $(\xi';\tau')=(3.5;0.01)$ (Figure (c)), $(\xi';\tau')=(5;0.01)$ (Figure (d)) and $(\xi';\tau')=(8;0.01)$ (Figure (e)). The space coordinate ξ is laid off along the abscissa, time variable τ – along the ordinate, and the values of the functions $G_j(\xi,\xi';\tau,\tau')$, $j=1,\ 2$ – along the applicate.

Note that the surfaces generated by the Green's functions (29) have a typical sharp peak in the vicinity of the point $(\xi;\tau)=(\xi';\tau')$ (see Figures 1 and 2). Here, in the vicinity of the layer boundary $\xi=0$ the values of the function $G_2(\xi,\xi';\tau,\tau')$ are times higher than $G_1(\xi,\xi';\tau,\tau')$ for the same values of the problem coefficients, for example, $\max\limits_{\xi,\tau\in K}G_2/\max\limits_{\xi,\tau\in K}G_1|_{(\xi';\tau')=(1;0.01)}=2.14$ (see Figures 1 (b) and 2 (b)). At the same time, near the boundary $\xi=\xi_0$ the difference

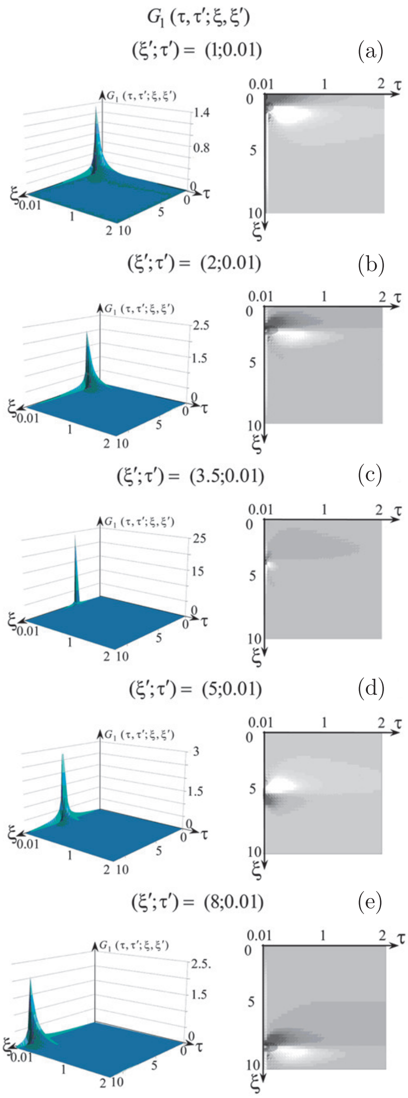


Figure 1. Surfaces of Green function $G_1(\xi,\xi';\tau,\tau')$ at different points $(\xi';\tau')$

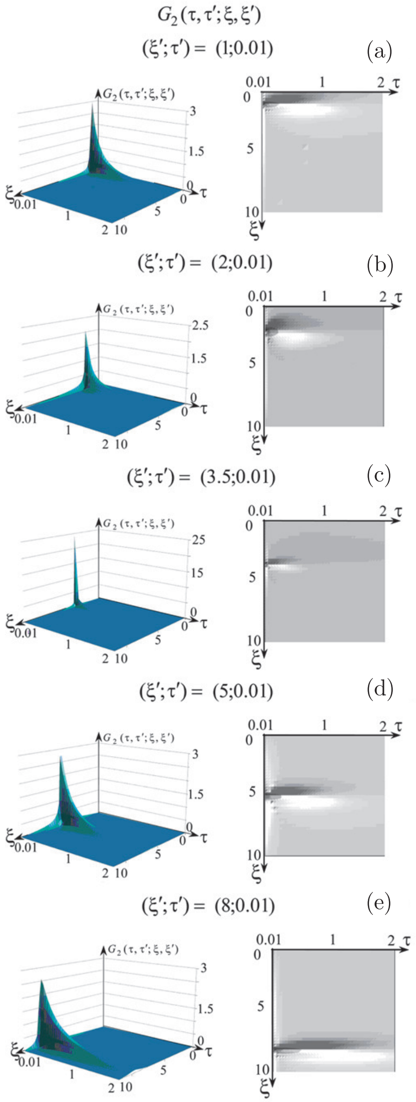


Figure 2. Surfaces of Green function $G_2(\xi,\xi';\tau,\tau')$ at different points $(\xi';\tau')$

riches 30%: $\max\limits_{\xi,\tau\in K} G_2 / \max\limits_{\xi,\tau\in K} G_1\big|_{(\xi';\tau')=(8;0.01)} = 1.29$, which is explained by a significantly greater sorption coefficient a_{11} than other "sorption" coefficients. Note that the values of $G_1(\xi,\xi';\tau,\tau')$ and $G_2(\xi,\xi';\tau,\tau')$ increase substantially with the shift of the coordinate of the point source ξ' to the middle of the layer, and their surfaces became flatter. In this case, a flatter descent is observed near the body boundary $\xi = 0$ for the function G_1 (see Figure 1 (a)), whereas such descent for the function G_2 is observed in the vicinity of the boundary $\xi = \xi_0$ (see Figure 2 (e)).

An increase in the coefficient a_{11} almost does not change the peak values of the functions $G_1(\xi,\xi';\tau,\tau')$ and $G_2(\xi,\xi';\tau,\tau')$, but somewhat slows down decreasing the Green's function along the time-axis $O\tau$.

The coefficient a_{21} variation in wide limits almost does not change the values of the function G_1, nor does it change the values of G_2 for small a_{11} (in this case the variation interval is contracted due to the physical constraints on the coefficients when the sorption processes are considered). For large values of the coefficient a_{11} growth of the coefficient a_{21} leads to an increase in the values of $G_2(\xi,\xi';\tau,\tau')$ and a flatter droop of its surface, *i.e.*, an increase in the time interval of the nonzero values of the Green's function.

For large values of the coefficient a_{22} ($a_{22} > 10$) there is a small domain of localization of nonzero values in the vicinity of the source of both Green's functions. A decrease in the values of a_{22} leads to an expansion of this region, moreover essentially for the function G_1 with small values of the coefficient a_{21} and for the function G_2 with large values of this coefficient. Note that the peak values of the function G_1 practically do not change with the change of the coefficient a_{21}, and an increase in the values of G_2 is typical with the growing values of a_{21}, for example, for the location of the source at the point $(\xi',\tau') = (5;0.01)$ $\max\limits_{\xi,\tau\in K'} G_1|_{a_{21}=79} / \max\limits_{\xi,\tau\in K'} G_1|_{a_{21}=2.2} = 1.07$ and $\max\limits_{\xi,\tau\in K'} G_2|_{a_{21}=79} / \max\limits_{\xi,\tau\in K'} G_2|_{a_{21}=2.2} = 1.52$ under $a_{22} = 1.1$.

3. Application of Green's functions to solving double-diffusivity problems

In the manufacture of complex composite structures in a number of technological processes, there is a need of gluing layers by certain substances. The problem to estimate the stability of the bonding layer arises during the period of use of structures of such type, Then there are two possible cases. The fact of the existence of such a layer is important in the process of the use of such a structure all the time, or it is desirable disappearance ("dissolution") of the gluing layer in the basic layers.

Moreover, lately the technologies of gluing of biological tissues have been increasingly used in complex medical operations. In this case it is advisable to prognosticate the resolution of the corresponding sutures.

In a mathematical description of the processes of mass transfer in the mentioned media (bimetals and alloys, composite materials, biological objects, soil, *etc.*) it is often necessary to take into consideration two ways of migration with significantly different diffusion coefficients and the mass exchange between them. Therefore, we consider some applications of the Green's functions obtained in the previous paragraph for the determination of solutions of the initial-boundary value problems of double-diffusivity in a layer with two migration ways, see [23].

3.1. Initial-boundary value problem of double-diffusivity from point source under zero boundary conditions

Let the admixture substance migrate in two ways in a layer with thickness z_0 from a point mass source and this source is located in the internal domain of the body at the point $z = z_*$ (Figure 3). Let the coefficient α determine the portion of the admixture substance that comes from the source into the quick way of diffusion (we assume that $d_0 > d > d_1 \geq d_2$).

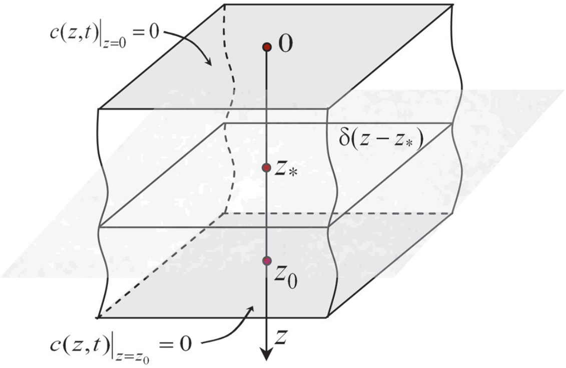


Figure 3. Layer in which the admixture migrates and a mass source acts at the point $z = z_*$

First, consider the case of zero initial and boundary conditions. Then the initial-boundary value problem on the elements of the vector-function $\mathbf{c}(z,t)$ normalized to the coefficient of diffusion in the quick way, can be formulated as

$$\begin{aligned} \frac{\partial c_1}{\partial t} - \frac{\partial^2 c_1}{\partial z^2} - d_1 \frac{\partial^2 c_2}{\partial z^2} + a_{11} c_1 - a_{12} c_2 &= \alpha \delta(z - z_*) \\ \frac{\partial c_2}{\partial t} - d_2 \frac{\partial^2 c_1}{\partial z^2} + d \frac{\partial^2 c_2}{\partial z^2} + a_{21} c_1 - a_{22} c_2 &= (1 - \alpha) \delta(z - z_*) \end{aligned} \tag{31}$$

$$c_1(z,t)|_{t=0} = c_2(z,t)|_{t=0} = 0 \tag{32}$$

$$c_1(z,t)|_{z=0} = 0, \qquad c_2(z,t)|_{z=0} = 0 \tag{33}$$

$$c_1(z,t)|_{z=z_0} = c_2(z,t)|_{z=z_0} = 0 \tag{34}$$

where α $(0 \leq \alpha \leq 1)$ is the parameter determining the part of the admixture substance that comes from the mass source into the quick way of migration (for example, in a water porous solution).

Note that we have changed to the "natural" dimensionless coordinates

$$z = (\tilde{a}_{12}/D_0)^{1/2} \xi, \qquad t = \tilde{a}_{12} \tau \tag{35}$$

Here $\tilde{a}_{12} = a_{12}$ if we do not take into consideration admixture decay processes or chemical reactions and $\tilde{a}_{12} \neq a_{12}$ (previously $\tilde{a}_{12} > a_{12}$) if we take into account certain physical, chemical or biological processes. The "natural" dimensionless form elongates the space coordinate axis and constricts the time axis that is important for the diffusion processes because the characteristic times of these processes are large in numerical terms and at the same time the values of the diffusion coefficients are very small. Note also that the "natural" dimensionless form does not use any geometrical parameters of the external or internal structure of the body. In this dimensionless form we have the problem coefficients

$$\begin{gathered} d_1 = D_1/D_0, \quad d_2 = D_2/D_0, \quad d = D/D_0 \\ a_{11} = \tilde{a}_{11}/\tilde{a}_{12}, \quad a_{21} = \tilde{a}_{21}/\tilde{a}_{12}, \quad a_{22} = \tilde{a}_{22}/\tilde{a}_{12}, \quad z_0 = (\tilde{a}_{12}/D_0)^{1/2}\xi_0 \end{gathered} \tag{36}$$

In the general case, using the Green functions, the solution of the problem (31)–(34) can be presented as follows, see [6]

$$\begin{gathered} c_1(z,t) = \alpha \int_0^t \int_0^{z_0} G_1(z,z';t,t')\delta(z-z_*)dz'dt' \\ c_2(z,t) = (1-\alpha) \int_0^t \int_0^{z_0} G_2(z,z';t,t')\delta(z-z_*)dz'dt' \end{gathered} \tag{37}$$

Substituting the expressions (29) into the formulae (37), after integrating we obtain

$$\begin{gathered} c_1(z,t) = \frac{2\alpha}{z_0}\sum_{n=1}^{\infty}\sin(y_n z)\sin(y_n z_*)\times \\ \left(\frac{A_1}{\eta_2} + \frac{1}{s_1 - s_2}\left[\left(1+\frac{A_1}{s_1}\right)e^{s_1 t} - \left(1+\frac{A_1}{s_2}\right)e^{s_2 t}\right]\right) \\ c_2(z,t) = \frac{2(1-\alpha)}{z_0}\sum_{n=1}^{\infty}\sin(y_n z)\sin(y_n z_*)\times \\ \left(\frac{A_2}{\eta_2} + \frac{1}{s_1 - s_2}\left[\left(1+\frac{A_2}{s_1}\right)e^{s_1 t} - \left(1+\frac{A_2}{s_2}\right)e^{s_2 t}\right]\right) \end{gathered} \tag{38}$$

Note that the functions $c_i(z,t)$ (38) are steadily increasing in the time variable (because s_1, $s_2 < 0$), and at the same time they are limited by certain (asymptotic) expressions in the steady-state regime

$$\begin{gathered} \lim_{t\to\infty} c_1(z,t) = \alpha\frac{2}{z_0}\sum_{n=1}^{\infty}\frac{\sin(y_n z)\sin(y_n z_*)}{\eta_2}A_1 \\ \lim_{t\to\infty} c_2(z,t) = (1-\alpha)\frac{2}{z_0}\sum_{n=1}^{\infty}\frac{\sin(y_n z)\sin(y_n z_*)}{\eta_2}A_2 \end{gathered} \tag{39}$$

A numerical analysis of the obtained solutions of the problem of double-diffusivity from the point source under zero initial and boundary conditions is performed for the following basic parameters of the problem: $d = 0.1$, $d_0 = 1$, $d_1 = d_2 = 0$, $a_{11} = 4$, $a_{12} = 1$, $a_{21} = 4$, $a_{22} = 1$, $z_0 = 10$. Figure 4 shows the typical distributions of the admixture substance $c_1(z,t)$ (dashed lines) in the quick way of migration, $c_2(z,t)$ (dash-and-dots lines) in the slow way and their sum $c(z,t)$ (full lines) for $\alpha = 0.25$ (Figure (a)), $\alpha = 0.5$ (Figure (b)), $\alpha = 0.75$ (Figure (c)) and $\alpha = 0.91$ (Figure (d)). Curves 1–5 correspond to such coordinates of disposition of the point source as $z_* = 1; 3; 5; 7; 9$.

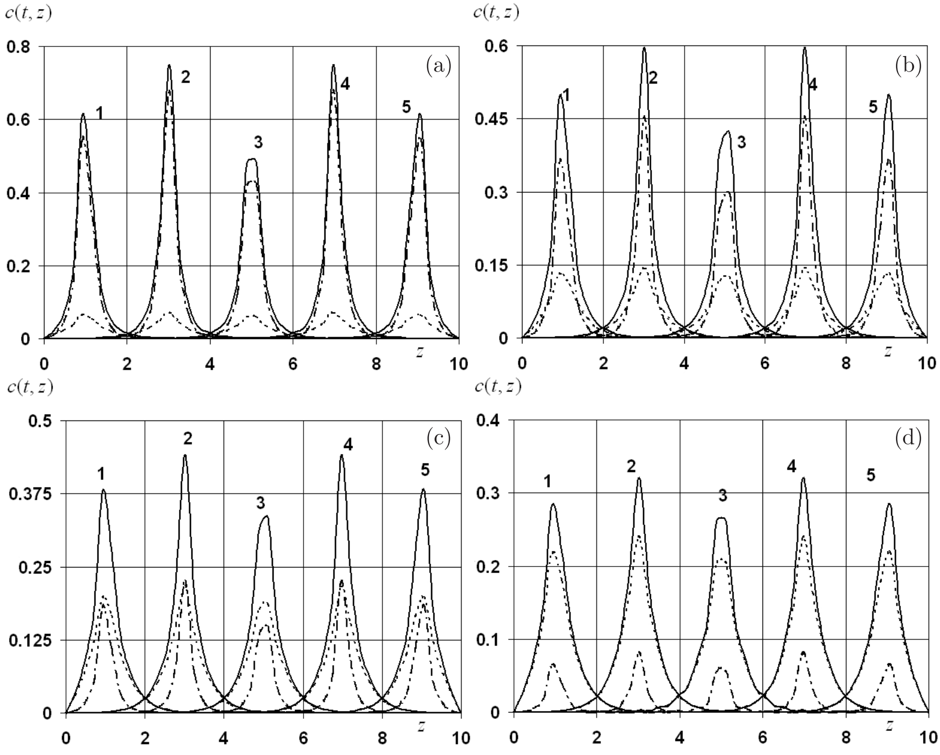


Figure 4. Distributions of admixture concentrations $c_1(z,t)$ and $c_2(z,t)$ in the in quick and slow migration ways, respectively, and their sum $c(z,t)$ for $\alpha = 0.25$ (a), $\alpha = 0.5$ (b), $\alpha = 0.75$ (c) and $\alpha = 0.91$ (d) for different point mass source coordinates

Note that both the particle concentrations in individual migration ways and the total concentration have the maximum values in the point source vicinity (see Figure 4). A shift in the point source coordinate deep into the body (to the bottom boundary of the layer) leads to similar shifts in the maximum values of the functions $c_1(z,t)$, $c_2(z,t)$ and $c(z,t)$, but the trajectories of motion of $\max_{z\in[0;z_0]} c_j(z,t)$ are of an oscillatory nature (see Figure 4). We also note that the coefficient determining the portion of the admixture substance that comes into the quick migration way from the source, affects significantly the spatial distributions of concentrations. Thus, the larger the value of α, the greater the admixture concentration in the quick way and the greater the contribution of $c_1(z,t)$ to the total concentration (see Figure 4 (d)). However, with increasing α, the total concentration values decrease significantly for all the point source coordinates (see Figures 4 (a)–4 (d)).

Figures 5–8 show the influence of the problem parameters on the distributions of the total concentration $c(z,t)$ for the same basic parameters of the problem and for the point source coordinate $z_* = 3$. Figure 5 illustrates the distributions of the total admixture concentration in different moments of time $t = 0.01; 0.1; 0.8; 2; 10; 100; 200$ (curves 1–7). Figure 6 shows the influence of the parameter α on the function $c(z,t)$, here the curves 1–5 correspond to the values $\alpha = 0.1; 0.25; 0.5; 0.75; 0.91$. In Figure 7 distributions of the total admixture concentration are given for different values of the reduced particle diffusion coefficient in the slow migration way $d = 0.1; 0.2; 0.5; 0.7; 0.95$ (curves 1–5, respectively). Figure 8 illustrates the typical distributions of the total admixture concentration $c(z,t)$ depending on different values of the "sorption" coefficient $a_{11} = 2; 4; 10; 20; 50$ (curves 1–5).

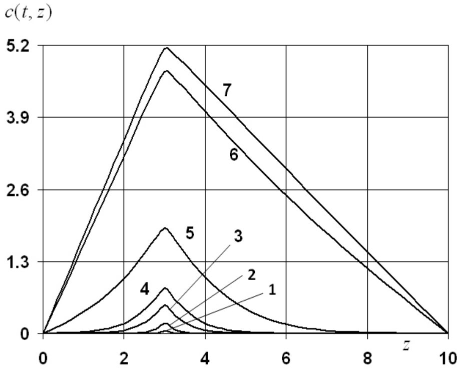


Figure 5. Distributions of total admixture concentration $c(z,t)$ in different moments of time

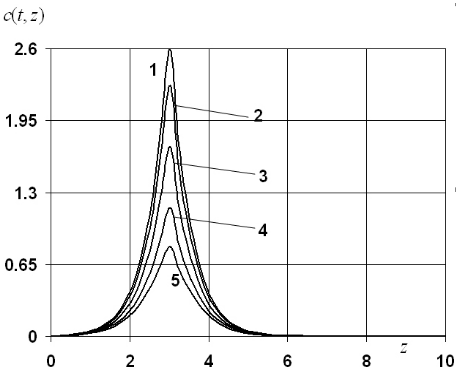


Figure 6. Distributions of total admixture concentration $c(z,t)$ for different values of parameter α

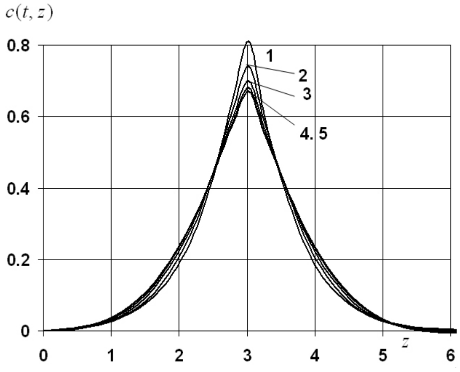


Figure 7. Distributions of total admixture concentration $c(z,t)$ for different values of reduced coefficient of diffusion d

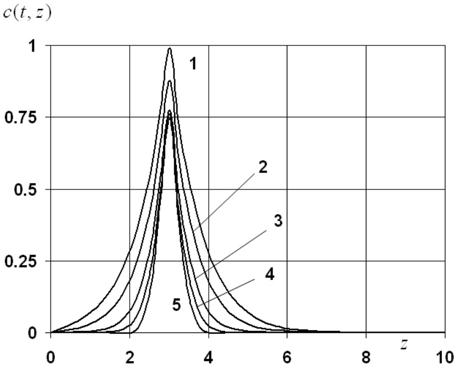


Figure 8. Distributions of total admixture concentration $c(z,t)$ for different values of "sorption" coefficient a_{11}

Note that the total admixture concentration in the body increases substantially in time, and its maximum remains at the point of action of the mass source (see Figure 5). For small times (curve 1 in Figure 5) a substantially nonlinear distribution of the total concentration is characteristic, and in the steady-state case (curve 7 in Figure 5) there are linear distributions in the intervals $[0; z_*)$ and $(z_*; z_0]$.

A change in the admixture distribution coefficient between states in the vicinity of action of the point source α does not change the behavior of function $c(z,t)$, but its influence on the total concentration values is great (see Figure 6). Changing the parameter α by 0.25 leads to a change in the maximum values of the concentration at the point of the point source by 35%, for example $\max\limits_{z\in[0;z_0]} c(z,t)|_{\alpha=0.1} / \max\limits_{z\in[0;z_0]} c_1(z,t)|_{\alpha=0.91} = 3.2$.

A change in the reduced diffusion coefficient in the slow way leads to changing the function of concentration in the vicinity of the point mass source action (see Figure 7). At the same time, an increase in the diffusion coefficient leads to a decrease in the maximum values of the total concentration. The coefficient of sorption a_{11} affects significantly the total concentration of the admixture particles that migrate in two ways from the point source. When this coefficient decreases, the total concentration increases throughout the body, and the drop of the function $c(z,t)$ from the source becomes flatter (see Figure 8).

3.2. Initial-boundary value problem of double-diffusivity from point source under support of constant value of total concentration at layer boundary

Now consider an analogical problem of double-diffusivity of the admixture substance in a layer with the support of a constant value of the total concentration at the body boundary $z=0$ (see Figure 9).

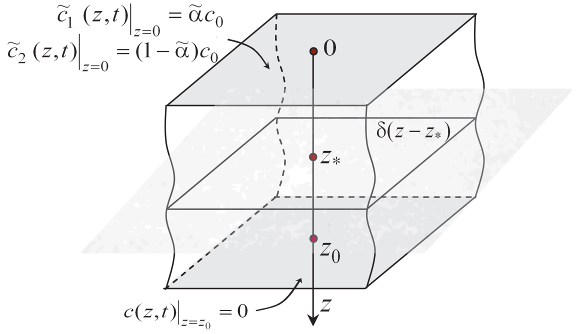


Figure 9. Layer in which admixture migrates and mass sources act in point $z=z_*$ and at boundary $z=0$

Such a process of mass transfer is described by the initial-boundary value problem (31), (32) and (34) under the following boundary condition at the upper surface of the body

$$\tilde{c}_1(z,t)|_{z=0} = \tilde{\alpha}c_0, \qquad \tilde{c}_2(z,t)|_{z=0} = (1-\tilde{\alpha})c_0 \tag{40}$$

where c_0 is the constant value of the total admixture concentration that is supported at the body boundary $z=0$; $\tilde{\alpha}$ $(0\le\tilde{\alpha}\le 1)$ is the parameter determining the portion of the admixture substance that comes from the body surface into the quick migration way; this coefficient can coincide with $(\tilde{\alpha}=\alpha)$ or differ $(\tilde{\alpha}\neq\alpha)$

from the parameter determining the portion of the admixture substance that comes into the quick way from the inner point mass source.

Taking into account the known expressions for the solutions of the double-diffusivity problem under the action of the constant source at the body surface, the solution of the problem (31), (32), (34) and (40) can be found by the formula

$$\frac{1}{c_0}\tilde{\mathbf{c}}(z,t)=\tilde{\mathbf{c}}^{(0)}(z,t)+\mathbf{A}\int_0^t\int_0^{z_0}\mathbf{G}(z,z';t,t')\delta(z-z_*)dz'dt' \tag{41}$$

where

$$\tilde{\mathbf{c}}^{(0)}(z,t)=\begin{pmatrix}\tilde{c}_1^{(0)}(z,t)\\ \tilde{c}_2^{(0)}(\xi,\tau)\end{pmatrix} \tag{42}$$

is the solution of the homogeneous problem of double-diffusivity under the initial and boundary conditions (32), (34) and (40), see [24]; and also

$$\mathbf{A}=\begin{pmatrix}\alpha/c_0\\ (1-\alpha)/c_0\end{pmatrix} \tag{43}$$

As a result we obtain

$$\begin{gathered}\frac{\tilde{c}_1(z,t)}{c_0}=\left\{\alpha-\frac{\tilde{b}_1}{ce}\right\}\left(1-\frac{z}{z_0}\right)-B\left[\left(\tilde{a}_1+\frac{\tilde{b}_1}{x_1}\right)\frac{\mathrm{sh}(\pi-y)x_1}{\mathrm{sh}\pi x_1}-\right.\\ \left.\left(\tilde{a}_1+\frac{\tilde{b}_1}{x_2}\right)\frac{\mathrm{sh}(\pi-y)x_2}{\mathrm{sh}\pi x_2}\right]-\frac{2}{z_0}\sum_{n=1}^{\infty}\frac{\sin(y_nz)}{s_1-s_2}\times\\ \left\{\frac{1}{y_n}\left[\left(\alpha s_1+p_1+\frac{p_2}{s_1}\right)e^{s_1t}-\left(\alpha s_2+p_1+\frac{p_2}{s_2}\right)e^{s_2t}\right]+\right.\\ \left.\alpha c_0\sin(y_nz_*)\left(\frac{A_1(s_1-s_2)}{\eta_2}+\left(1+\frac{A_1}{s_1}\right)e^{s_1t}-\left(1+\frac{A_1}{s_2}\right)e^{s_2t}\right)\right\}\end{gathered} \tag{44}$$

$$\begin{gathered}\frac{\tilde{c}_2(z,t)}{c_0}=\left\{1-\alpha-\frac{\tilde{b}_2}{ce}\right\}\left(1-\frac{z}{z_0}\right)-B\left[\left(\tilde{a}_2+\frac{\tilde{b}_2}{x_1}\right)\frac{\mathrm{sh}(\pi-y)x_1}{\mathrm{sh}\pi x_1}-\right.\\ \left.\left(\tilde{a}_2+\frac{\tilde{b}_2}{x_2}\right)\frac{\mathrm{sh}(\pi-y)x_2}{\mathrm{sh}\pi x_2}\right]-\frac{2}{z_0}\sum_{n=1}^{\infty}\frac{\sin(y_nz)}{s_1-s_2}\times\\ \left\{\frac{1}{y_n}\left[\left((1-\alpha)s_1+p_1'+\frac{p_2'}{s_1}\right)e^{s_1t}-\left((1-\alpha)s_2+p_1'+\frac{p_2'}{s_2}\right)e^{s_2t}\right]+\right.\\ \left.(1-\alpha)c_0\sin(y_nz_*)\left(\frac{A_2(s_1-s_2)}{\eta_2}+\left(1+\frac{A_2}{s_1}\right)e^{s_1t}-\left(1+\frac{A_2}{s_2}\right)e^{s_2t}\right)\right\}\end{gathered} \tag{45}$$

$$
\begin{aligned}
\frac{\tilde{c}(z,t)}{c_0} = {} & \left\{1-\frac{\tilde{b}_1+\tilde{b}_2}{ce}\right\}\left(1-\frac{z}{z_0}\right)-B\left[\left(\tilde{a}_1+\frac{\tilde{b}_1}{x_1}\right)\left(\tilde{a}_2+\frac{\tilde{b}_2}{x_1}\right)\frac{\mathrm{sh}(\pi-y)x_1}{\mathrm{sh}\pi x_1}-\right. \\
& \left.\left(\tilde{a}_1+\frac{\tilde{b}_1}{x_2}\right)\left(\tilde{a}_2+\frac{\tilde{b}_2}{x_2}\right)\frac{\mathrm{sh}(\pi-y)x_2}{\mathrm{sh}\pi x_2}\right]-\frac{2}{z_0}\sum_{n=1}^{\infty}\frac{\sin(y_n z)}{s_1-s_2}\left\{\frac{1}{y_n}\times\right. \\
& \left[\left((1-\alpha)s_1+p_1+p_1'+\frac{p_2+p_2'}{s_1}\right)e^{s_1 t}-\left((1-\alpha)s_2+p_1+p_1'+\frac{p_2+p_2'}{s_2}\right)e^{s_2 t}\right]+ \\
& \left. c_0\sin(y_n z_*)\left(A_\alpha\frac{s_1-s_2}{\eta_2}+\left[1+\frac{A_\alpha}{s_1}\right]e^{s_1 t}-\left[1+\frac{A_\alpha}{s_2}\right]e^{s_2 t}\right)\right.
\end{aligned}
\tag{46}
$$

where $A_\alpha=\alpha A_1+(1-\alpha)A_2$, $p_1=(\alpha d-d_1(1-\alpha))y_n^2+\alpha a_{22}+\alpha_1+(1-\alpha)a_{12}$, $p_2(d_1\alpha_2+d\alpha_1)y_n^2+\alpha_1 a_{22}-\alpha a_{12}$, $p_1'=(1-\alpha-\alpha d_2)y_n^2+(1-\alpha)a_{11}+\alpha a_{21}+\alpha_2$, $p_2'=(\alpha_2+\alpha_1 d_2)y_n^2+\alpha_2 a_{11}-\alpha_1 a_{21}$, $B=1/\sqrt{d^2-4ec^2}$, $x_{1,2}=\left(-d/c\pm\sqrt{d^2-4ec^2}\right)/2$, $y=\pi z/z_0$, $b=\alpha_1 a_{22}-\alpha_2 a_{12}$, $a=(d_1\alpha_2+d\alpha_1)\pi^2/z_0^2$, $c=(d-d_1 d_2)\pi^4/z_0^4$, $d=(a_{22}+a_{12}d+d_1 a_{21}+d_2 a_{12})\pi^2/z_0^2$, $e=a_{11}a_{21}-a_{12}a_{21}$, $\tilde{a}_1=(d_1\alpha_2+d\alpha_1)\pi^2/z_0^2$, $\tilde{a}_2=(\alpha_2+\alpha_1 d_2)\pi^2/z_0^2$, $\tilde{b}_1=\alpha_1 a_{22}-\alpha_2 a_{12}$, $\tilde{b}_2=\alpha_2 a_{11}-\alpha_1 a_{21}$, $\alpha_1=a_{12}(1-\alpha)-a_{11}\alpha$, $\alpha_2=a_{22}(1-\alpha)-a_{12}\alpha$.

A numerical analysis of the obtained solutions of the problem of double-diffusivity from the point source under the second constant mass source at the layer boundary $z=0$, *i.e.* formulaes (44)–(46), is performed for the basic parameters indicated in Subsection 3.1. Note that the calculations are carried out for a case where the mass transfer process is accompanied by the processes of sorption-desorption only, *i.e.* $a_{11}=a_{21}$ and $a_{12}=a_{22}$. At this subsection, we have $a=a_{11}/a_{12}=3$ as a basic value.

Figure 10 illustrates the typical distributions of the admixture substance $\tilde{c}_1(z,t)/c_0$ (dashed lines, curves a) in the quick way of migration, $\tilde{c}_2(z,t)/c_0$ (dash-and-dots lines, curves b) in the slow way and their sum $\tilde{c}(z,t)/c_0$ (full lines) for $\tilde{\alpha}=0$ (Figure (a)) and $\tilde{\alpha}=0.91$ (Figure (b)). Curves 1 correspond to the value of the diffusion coefficient $d=0.1$, curves 2 – $d=0.01$.

Figures 11–14 show the influence of the problem parameters on the distributions of the total concentration $\tilde{c}_1(z,t)$ for the same basic parameters of the problem and for the coordinate of the point source $z_*=3$ for $\tilde{\alpha}=0$ (Figure (a)) and $\tilde{\alpha}=0.91$ (Figure (b)). Figure 11 illustrates the distributions of the total concentration of the admixture $\tilde{c}_1(z,t)/c_0$ in different moments of time $t=0.5;1;2;10;20$ (curves 1–5). Figure 12 shows the influence of the parameter α on the function $\tilde{c}_1(z,t)/c_0$, here curves 1–5 correspond to the values $\alpha=0.1;0.25;0.5;0.75;0.91$. In Figure 13 the distributions of the total admixture concentration are given for different values of the reduced coefficient of the sorption intensity $a_{11}=0.8;2;3;5$ (curves 1–5, respectively). Figure 14 illustrates the typical distributions of the total admixture concentration depending on different values of the intensity of the point mass source $\tilde{c}_0=1/c_0=2;1.5;1;0.5;0.25;0.1$ (curves 1–5).

Note that the occurrence of two mass sources (the constant one at the boundary $z=0$ and the point source at the inner point $z_*=3$) influences

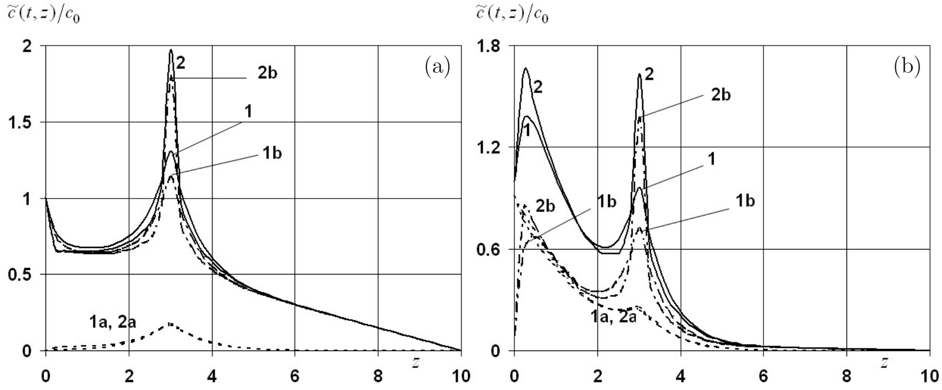


Figure 10. Distributions of admixture concentration $\tilde{c}_1(z,t)/c_0$ in the quick migration way, $\tilde{c}_2(z,t)/c_0$ in the slow way and their sum $\tilde{c}(z,t)/c_0$ for different values of diffusion coefficient $d = 0.1$ (curves 1) and 0.01 (curves 2) under $\tilde{\alpha} = 0$ (a) and $\tilde{\alpha} = 0.91$ (b)

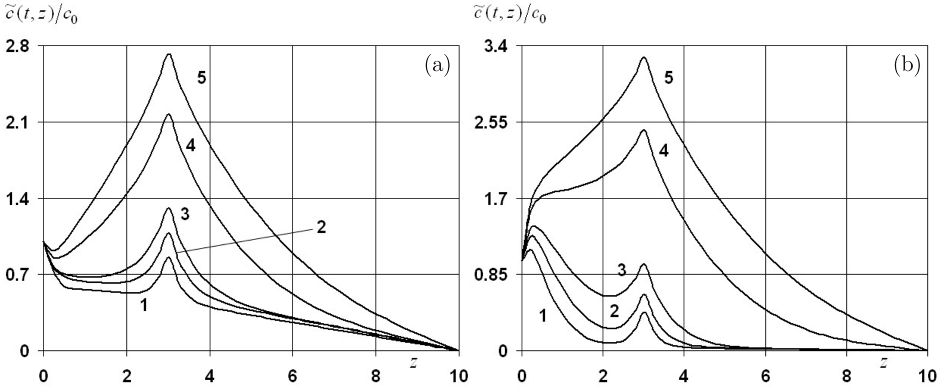


Figure 11. Distributions of total admixture concentration $\tilde{c}(z,t)/c_0$ in different moments of time under $\tilde{\alpha} = 0$ (a) and $\tilde{\alpha} = 0.91$ (b)

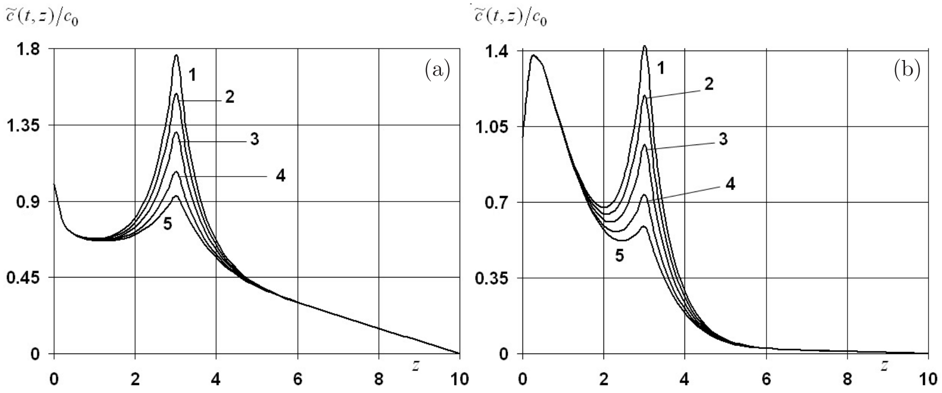


Figure 12. Distributions of total admixture concentration $\tilde{c}(z,t)/c_0$ for different values of the coefficient α under $\tilde{\alpha} = 0$ (a) and $\tilde{\alpha} = 0.91$ (b)

substantially the admixture concentration. In this case, for the predominant penetration of particles from the body surface in the slow diffusion way, *i.e.* $\alpha \sim 0$ (see Figures 10 (a)–14 (a)), the function $\tilde{c}(z,t)/c_0$ near the surface is decreasing, and in the absence of a point source it is steadily decreasing in the whole interval, see [24]. If the admixture falls mainly from the surface in the quick way of diffusion, *i.e.* $\tilde{\alpha} = 0.91$ (see Figures 10 (b)–14 (b)), there is always the subsurface maximum of the total concentration.

For small $\tilde{\alpha}$ the admixture concentration in the slow way contributes mainly in the total concentration (curves 1b and 2b in Figure 10 (a) correspondingly). For large $\tilde{\alpha}$ in the vicinity of the surface $z = 0$ concentrations $\tilde{c}_1(z,t)/c_0$ and $\tilde{c}_2(z,t)/c_0$ are commensurable, whereas near the point mass source the values of concentration in the slow way increase substantially (curves 1a and 2b in Figure 10 (b) correspondingly). Also we note that the smaller the ratio of the diffusion coefficients d, the larger the values of the total concentration reaches independently on the coefficient $\tilde{\alpha}$ (curves 2 in Figures 10 (a) and 10 (b)).

For small time intervals the constant source at the layer boundary makes a predominant contribution in the values of $\tilde{c}(z,t)/c_0$. With an increase in time of the double-diffusivity process, the total admixture concentration increases throughout the body and an increasing contribution is made by the point mass source (see Figure 11).

A change in the coefficient α, *i.e.* the part of the admixture substance that comes from the point source into the body, does not change the behavior of the function of the total concentration, but affects significantly its values in the vicinity of the point source (see Figure 12). Hence, the smaller the values the coefficient $\alpha \in [0,1]$ takes up, the larger the values the local maximum $\max\limits_{z=z_*} \tilde{c}(z,t)/c_0$ reaches. And, for example, for $\tilde{\alpha} = 0.91$ (curves 1 in Figures 12 (a) and 12 (b)) this maximum becomes global for any α.

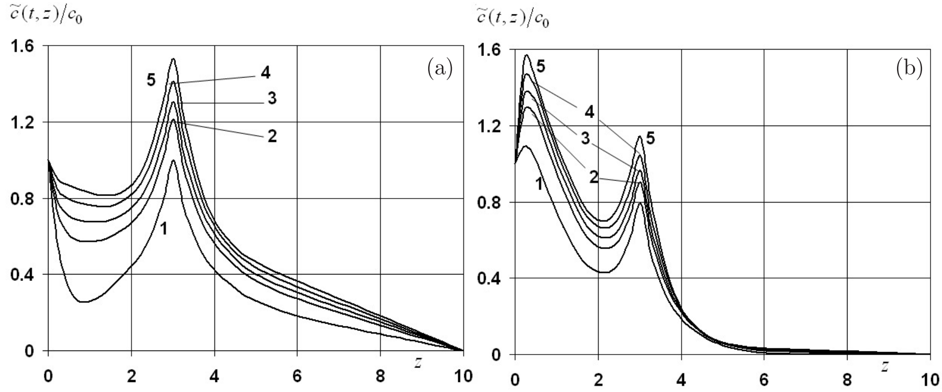


Figure 13. Distributions of total admixture concentration $\tilde{c}(z,t)/c_0$ for different values of coefficient a_{11} under $\tilde{\alpha} = 0$ (a) and $\tilde{\alpha} = 0.91$ (b)

Note that the ratio of the “sorption intensity” coefficient to the “desorption intensity” coefficient a_{11} affects essentially the distribution of the total concentration. Moreover, the larger the coefficient a_{11}, the greater the value that the total concentration takes up (see Figure 13). For large $\tilde{\alpha}$ and any a_{11} the subsurface maximum is always global (see Figure 13 (b)), and for large values of a_{11}, $\max\limits_{z\in[0;z_0/10]} \tilde{c}(z,t) \Big/ \max\limits_{z=z_*} \tilde{c}(z,t) \Big|_{a_{11}=10} = 1.36$, for small values of the coefficient a_{11} $\max\limits_{z\in[0;z_0/10]} \tilde{c}(z,t) \Big/ \max\limits_{z=z_*} \tilde{c}(z,t) \Big|_{a_{11}=10} = 1.38.$

In the case of predominant penetration of the admixture particles from the body boundary into the slow migration way, the maximum forming near the point mass source is global under a more intensive process of sorption than desorption (curves 2–5 in Figure 13 (a)). In the opposite case, *i.e.* under a more intensive desorption process, this maximum of the function $\tilde{c}(z,t)$ is local (curve 1 in Figure 13 (a)).

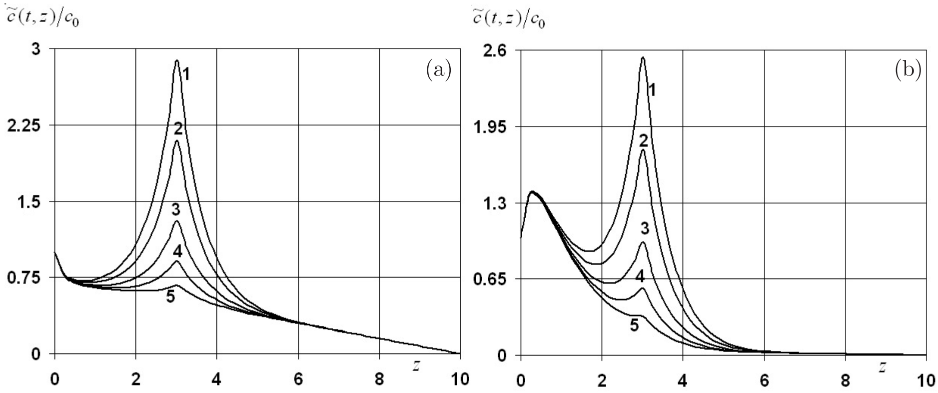


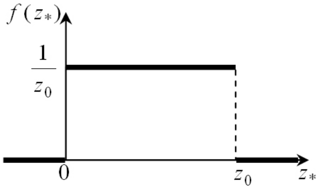
Figure 14. Distributions of total admixture concentration $\tilde{c}(z,t)/c_0$ for different values of concentration at body boundary c_0 under $\tilde{\alpha}=0$ (a) and $\tilde{\alpha}=0.91$ (b)

With the increasing intensity of the point source, *i.e.*, an increase in the value of $\tilde{c}_0 = 1/c_0$, the total admixture concentration in the vicinity of the point mass source grows substantially (see Figure 14). And for $\tilde{c}_0 > 0.5$ $\max\limits_{z=z_*} \tilde{c}(z,t)$ is global (curves 1–3 in Figure 14) and the domain of influence of the point mass source on the total admixture migrating in two ways enlarges.

3.3. Initial-boundary value problem of double-diffusivity from random point source

Uniform distribution.

Consider the problems of double-diffusivity of the admixture substance in a layer under the action of random point mass sources. We assume that the coordinate of location of the mass source is unknown, *i.e.* the quantity z_* is random (see Figure 3). Let a uniform distribution of the random quantity be given in the interval $(0,z_0)$:



$$f(z_*)=\begin{cases}\frac{1}{z_0}, & z_*\in(0,z_0),\\ 0, & z_*\notin(0,z_0)\end{cases} \tag{47}$$

Then such process is described by the system of equations (31). And we consider the case of zero initial and boundary conditions (32)–(34).

The solution of this problem is obtained by averaging the expressions (38) with respect to the uniform distribution function, see [25]. Then, we have

$$\begin{gathered}\bar{c}_1(z,t)=\langle c_1(z,t)\rangle_{z_*}=\alpha\frac{2}{z_0^2}\sum_{n=1}^{\infty}\frac{\sin(y_n z)}{y_n}B_n\times\\ \left(\frac{A_1}{\eta_2}+\frac{1}{s_1-s_2}\left[\left(1+\frac{A_1}{s_1}\right)e^{s_1t}-\left(1+\frac{A_1}{s_2}\right)e^{s_2t}\right]\right)\\ \bar{c}_2(z,t)=\langle c_2(z,t)\rangle_{z_*}=(1-\alpha)\frac{2}{z_0^2}\sum_{n=1}^{\infty}\frac{\sin(y_n z)}{y_n}B_n\times\\ \left(\frac{A_2}{\eta_2}+\frac{1}{s_1-s_2}\left[\left(1+\frac{A_2}{s_1}\right)e^{s_1t}-\left(1+\frac{A_2}{s_2}\right)e^{s_2t}\right]\right)\\ \bar{c}(z,t)=\langle c(z,t)\rangle_{z_*}=\frac{2}{z_0^2}\sum_{n=1}^{\infty}\frac{\sin(y_n z)B_n}{y_n}\times\\ \left(\frac{A_\alpha}{\eta_2}+\frac{1}{s_1-s_2}\left[\left(1+\frac{A_\alpha}{s_1}\right)e^{s_1t}-\left(1+\frac{A_\alpha}{s_2}\right)e^{s_2t}\right]\right)\end{gathered} \tag{48}$$

where $B_n=1-(-1)^n$.

Figure 15 illustrates typical distributions of the averaged total concentration of the admixture $\bar{c}(z,t)$ under a uniform distribution of the point mass source coordinate in different moments of time $t=0.5;1;2;10;15;20$ (curves 1–6) for $\alpha=0.25$. Figure 16 shows the influence of the parameter α at the moment $t=2$ on the function $\bar{c}(z,t)$, here the curves 1–5 correspond to the values $\alpha=0.1;0.25;0.5;0.75;0.91$. In Figure 17 the distributions of the averaged total admixture concentration are given for different values of the reduced coefficient of sorption and desorption intensities $a_{11}=0.8;2;3;5;10$ (curves 1–5 respectively) for small $t=2$ (Figure (a)) and large $t=20$ (Figure (b)) times.

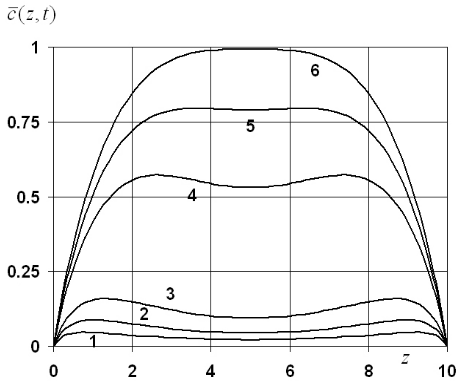


Figure 15. Distributions of averaged total concentration of admixture $\bar{c}(z,t)$ under uniform source coordinate distribution at different moments

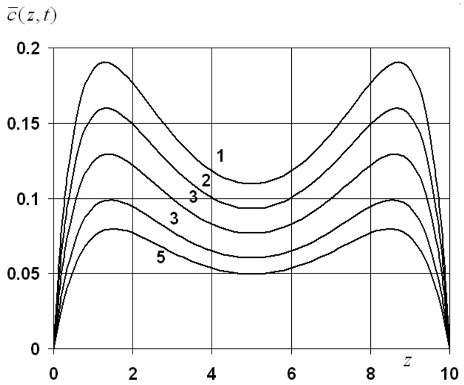


Figure 16. Distributions of averaged total concentration of admixture $\bar{c}(z,t)$ under uniform source coordinate distribution for different values of coefficient α

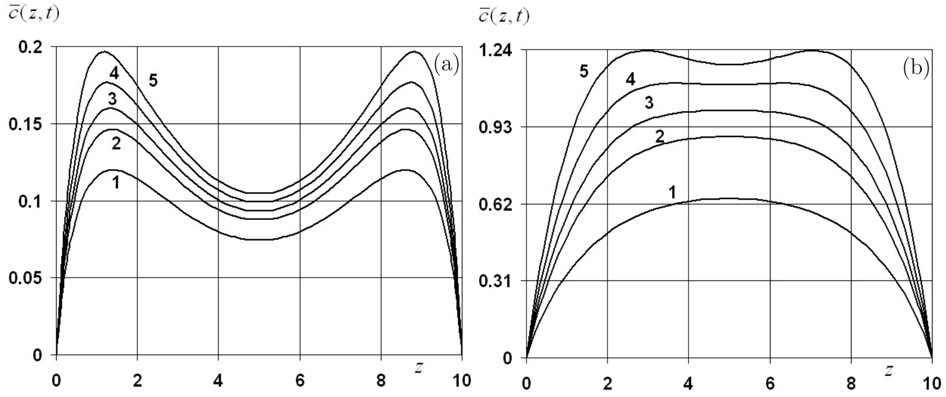


Figure 17. Graphs of averaged total concentration of admixture $\tilde{c}(z,t)/c_0$ under uniform source coordinate distribution depending on different values of coefficient a_{11} for small (a) and large (b) times

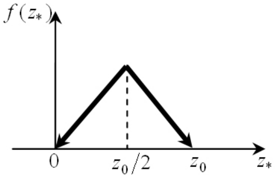
Note that under a uniform distribution of a random location of the point mass source, for small times, the averaged total concentration of admixture particles has two maxima of the same values at about the points $z^1_{max} = z_0/7$ and $z^2_{max} = z_0 - z_0/7$ (see Figure 15) (48). The local minimum of function $\bar{c}(z,t)$ disposes at the point $z_{min} = z_0/2$. As the time of the double-diffusivity process increases, the averaged total concentration increases throughout the layer. And the difference between maximal values and local minimum decreases gradually, evens and a single maximum of the function $\bar{c}(z,t)$ is formed in the middle of the body for times close to the steady-state (curve 6 in Figure 15).

A change in the coefficient α determining the part that comes from the source into the quick migration way, influences only the values of the averaged total concentration, without changing its behavior (see Figure 16). Moreover, the smaller the coefficient α takes up the value, the larger the averaged concentration is, and $\max\limits_{z\in[0,z_0]}\bar{c}(z,t)\Big|_{\alpha=0}\Big/\max\limits_{z\in[0,z_0]}\bar{c}(z,t)\Big|_{\alpha=0.91}=1.64$. At the same time $\min\limits_{z\in[0,z_0]}\bar{c}(z,t)\Big|_{\alpha=0}\Big/\min\limits_{z\in[0,z_0]}\bar{c}(z,t)\Big|_{\alpha=0.91}=1.4$ (curves 1 and 5 in Figure 16).

The growth of the intensity of the sorption process relative to the desorption one, *i.e.* an increase in the coefficient a_{11}, leads to an increase in the averaged admixture concentration throughout the body for both small (see Figure 17 (a)) and large (see Figure 17 (b)) times. However, the larger the coefficient a_{11}, the later the averaged concentration reaches the steady-state regime (curves 5 and 1 in Figure 17 (b)). Note that the change in the ratio of the diffusion coefficients in different migration ways has little effect on the values of the averaged total concentration. Thus, a change in the coefficient d by an order leads to changes in the values of function $\bar{c}(z,t)$ to 2% for small time intervals and up to 3% for large ones.

Triangular distribution.

Let the coordinate of location of the point mass source z_* be a random quantity with a triangular distribution in the interval $(0,z_0)$. The function of the triangular distribution density (see [25]) is of the form of



$$f(z_*)=\begin{cases}\frac{2}{z_0}-\frac{2|z_0-2z_*|}{z_0^2}, & z_*\in(0,z_0),\\ 0, & z_*\notin(0,z_0)\end{cases} \tag{49}$$

We average the solution of the double-diffusivity problem (31)−(34) in the interval $(0,z_0)$ with respect to z_* with the distribution function (49). As a result we obtain

$$\bar{\bar{c}}_1(z,t)=\left\langle c_1(z,t)\right\rangle_{z_*}=\alpha\frac{8}{z_0^3}\sum_{n=1}^{\infty}\frac{\sin(y_nz)}{y_n}\bar{B}_n\times \left(\frac{A_1}{\eta_2}+\frac{1}{s_1-s_2}\left[\left(1+\frac{A_1}{s_1}\right)e^{s_1t}-\left(1+\frac{A_1}{s_2}\right)e^{s_2t}\right]\right) \tag{50}$$

$$\bar{\bar{c}}_2(z,t)=\left\langle c_2(z,t)\right\rangle_{z_*}=(1-\alpha)\frac{8}{z_0^3}\sum_{n=1}^{\infty}\frac{\sin(y_nz)}{y_n}\bar{B}_n\times \left(\frac{A_2}{\eta_2}+\frac{1}{s_1-s_2}\left[\left(1+\frac{A_2}{s_1}\right)e^{s_1t}-\left(1+\frac{A_2}{s_2}\right)e^{s_2t}\right]\right) \tag{51}$$

$$\bar{\bar{c}}(z,t)=\left\langle c(z,t)\right\rangle_{z_*}=\frac{8}{z_0^3}\sum_{n=1}^{\infty}\frac{\sin(y_nz)\bar{B}_n}{y_n}\times \left(\frac{A_\alpha}{\eta_2}+\frac{1}{s_1-s_2}\left[\left(1+\frac{A_\alpha}{s_1}\right)e^{s_1t}-\left(1+\frac{A_\alpha}{s_2}\right)e^{s_2t}\right]\right) \tag{52}$$

Here $\bar{B}_n = (-1)^{[n/2]}\{1-(-1)^n\}$, [] is the integer part of the number.

Note that the admixture concentrations $\bar{\bar{c}}_i$ $(i = 1,2)$, and accordingly the total concentration, in the layer under the action of the randomly disposed mass source with a triangular distribution of its coordinate in the domain differ from the corresponding functions $\bar{c}_i$ and $\bar{c}$ averaged with respect to z_* under a uniform distribution by the coefficient $1/z_0$ and by the multipliers $\bar{B}_n$, $n = 1,2,\ldots$. Considering that $1-(-1)^n$ equals 0 for even n, the number of computational procedures in a numerical analysis of the solutions (44)–(46) and (50)–(52) is reduced twice. In addition, the averaging of the solutions with respect to the quantity z_* leads to the asymptotic expressions (44)–(46) (concentrations in the steady-state regime) different from (50)–(52).

Figure 18 illustrates typical distributions of the averaged total concentration of the admixture $\bar{\bar{c}}(z,t)$ under a triangular distribution of the mass source coordinate in different moments of time $t = 0.5; 1; 2; 10; 15; 20$ (curves 1–6) for $\alpha = 0.25$. Figure 19 shows the influence of the parameter α at the moment $t = 2$ on the function $\bar{\bar{c}}(z,t)$, here the curves 1–5 correspond to the values $\alpha = 0.1; 0.25; 0.5; 0.75; 0.91$. In Figure 20 distributions of the averaged total admixture concentration are given for different values of the reduced coefficient of sorption and desorption intensities $a_{11} = 0.8; 2; 3; 5; 10$ (curves 1–5 respectively) for small $t = 2$ (Figure (a)) and large $t = 20$ (Figure (b)) times.

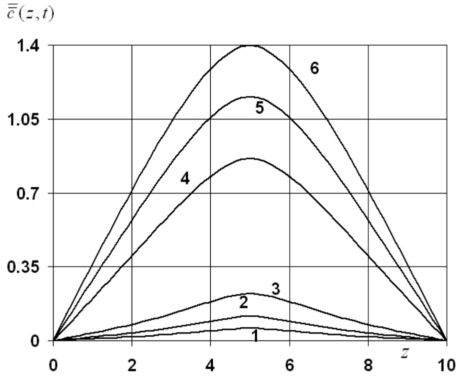


Figure 18. Graphs of averaged total concentrations of admixture substance $\bar{\bar{c}}(z,t)$ under source coordinate triangular distribution in different moments of time

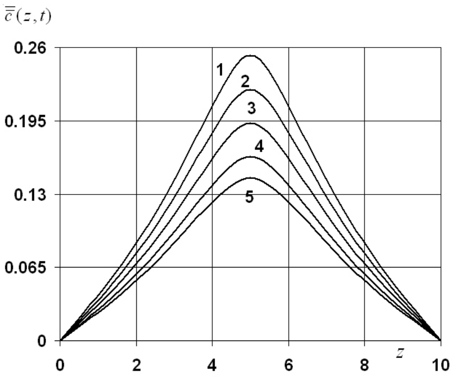


Figure 19. Graphs of averaged total concentrations of admixture substance $\bar{\bar{c}}(z,t)$ under source coordinate triangular distribution depending on values of coefficient α

Note that for a triangular distribution of the coordinate of the point mass source, as distinct from the uniform one, formation of a single maximum of the averaged admixture concentration occurs from the beginning of the double-diffusivity process (see Figure 18). Moreover, such a maximum is formed in the middle of the body, which corresponds to the given condition of the most probable location of the source exactly at the point $z = z_0/2$ (50)–(52). As the process time

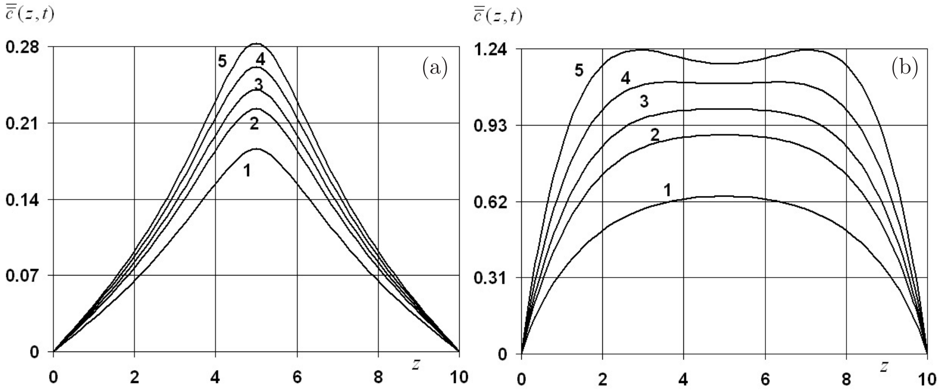


Figure 20. Graphs of averaged total concentrations of admixture substance $\bar{\bar{c}}(z,t)$ under source coordinate triangular distribution depending on values of coefficient a_{11} for small (a) and large (b) times

increases, the values of the averaged total concentration increase substantially in the middle of the layer (see Figure 18), with such behavior of the function $\bar{\bar{c}}(z,t)$ tends to parabolic.

The influence of the coefficient α determining the portion of the admixture that comes from the source into the quick diffusion way, is felt only on the values of the averaged concentration. Thus, with a decrease in the coefficient α, the admixture concentration $\bar{\bar{c}}(z,t)$ increases, and

$$\max_{z\in[0,z_0]} \bar{\bar{c}}(z,t)\Big|_{\alpha=0.1} \Big/ \max_{z\in[0,z_0]} \bar{\bar{c}}(z,t)\Big|_{\alpha=0.91} = 1.75 \text{ (see Figure 19).}$$

For a triangular distribution of the point source coordinate, as well as for the uniform one, the growth in the intensity of the sorption process relative to the desorption one leads to an increase in the averaged admixture concentration throughout the body for both small (see Figure 20 (a)) and large (Figure 20 (b)) times. In this case, the growing intensity of the sorption process also slows down reaching the steady-state regime by the function $\bar{\bar{c}}(z,t)$.

Note that for a triangular distribution of the source location the change in the ratio of the diffusion coefficients in different migration ways d, as for the uniform distribution, has little effect on the values of the averaged total concentration. However, the difference in the values of the function $\bar{\bar{c}}(z,t)$ is somewhat higher, in particular, when the coefficient d changes by an order the values of the averaged total concentration of the admixture change to 3% for small times and up to 8% for large ones.

4. Conclusion

The matrix Green's function under initial and boundary conditions of the first kind is defined for the second-order system of equations of double-diffusivity presented in the matrix form. The formulae for elements of the matrix Green's function are obtained and the behavior of the Green's functions is investigated. It is established that a quantitative relation between the Green's functions in different states is determined by the ratio of coefficients of the "sorption-desorption" type. At the same time the ratio of the diffusion coefficients in different migration ways and the time of running the double-diffusivity process are substantially influenced by the rate of the drop of the Green's functions down to the values in the vicinity of zero.

The obtained expressions for the Green's functions are used to find the solutions of the initial-boundary value problems of the double-diffusivity of the admixture substance under the action of the inner point mass source. We considered cases of both deterministic and stochastic sources under uniform and triangular distributions of the source location coordinate. It is established that the averaging of the function of the admixture concentration with respect to the point mass source coordinate leads to a reduction twice the number of the computing procedures for a numerical analysis of the solutions of the problems of the double-diffusivity of admixture particles.

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